Quantifying Double McCormick

Emily Speakman and Jon Lee
University of Michigan
{eespeakm, jonxlee}@umich.edu

Abstract: When using the standard McCormick inequalities twice to convexify trilinear monomials, as is often the practice in modeling and software, there is a choice of which variables to group first. For the important case in which the domain is a nonnegative box, we calculate the volume of the resulting relaxation, as a function of the bounds defining the box. In this manner, we precisely quantify the strength of the different possible relaxations defined by all three groupings, in addition to the trilinear hull itself. As a by product, we characterize the best double-McCormick relaxation.

We wish to emphasize that, in the context of spatial branch-and-bound for factorable formulations, our results do not only apply to variables in the input formulation. Our results apply to monomials that involve auxiliary variables as well. So, our results apply to the product of any three (possibly complicated) expressions in a formulation.

Key words: global optimization, mixed-integer nonlinear programming, spatial branch-and-bound, convexification, bilinear, trilinear, McCormick inequalities
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1. Introduction. Spatial branch-and-bound (sBB) (see [1, 20, 25]) for so-called factorable mathematical-optimization formulations (see [15]) is the workhorse general-purpose algorithm in the area of global optimization. It works by using additional variables to reformulate every function of the formulation as a (labeled) directed acyclic graph (DAG). Root nodes can be very complicated functions, and leaves are variables that appear in the input formulation, each labeled with its interval domain. Intermediate nodes are labeled with auxiliary variables together with operators from a small dictionary of basic functions of few (often one or two) variables. Also, we have a method for convexifying the graph of each dictionary function. sBB algorithms work by composing convex relaxations of the dictionary functions, according to the DAG, to get relaxations of the root functions. Bounds on the leaves propagate to other nodes and conversely. Branching (subdividing the domain interval of a variable) creates subproblems, which are treated recursively. Objective bounds for subproblems are appropriately combined to achieve a global-optimization algorithm.

Much of the research on sBB has focused on developing tight convexifications for basic functions of few variables (many references can be found in [4]). Other research has focused on how bounds can be efficiently propagated and how branching can be judiciously be carried out (see [3], for example). From the viewpoint of good convexifications, much less attention has been paid to how the DAGs are created, but this can have a strong impact on the quality of the resulting convex relaxation of the input formulation; see [12, 13, 22, 29] for some key papers with other viewpoints concerning constructing DAGs.

For basic multilinear monomials $f(x_1,\ldots,x_n) := x_1 \cdots x_n$, with $x_i \in [a_i, b_i]$, there is already a lot of flexibility which can have a significant impact on the overall convexification of the graph of $f(x_1,\ldots,x_n) := x_1 \cdots x_n$ on the box domain $[a_1,b_1] \times \cdots \times [a_n,b_n]$. For $n = 2$, we have the classic McCormick inequalities (see [15]), which is simply the tetrahedron that is the convex hull of

the points $(f, x_1, x_2) := (a_1a_2, a_1, a_2), (a_1b_2, a_1, b_2), (b_1a_2, b_1, a_2), (b_1b_2, b_1, b_2)$. The inequalities can be derived from the four inequalities

$$(x_1 - a_1)(x_2 - a_2) \geq 0,\ (x_1 - a_1)(b_2 - x_2) \geq 0,$$
$$(b_1 - x_1)(x_2 - a_2) \geq 0,\ (b_1 - x_1)(b_2 - x_2) \geq 0,$$

by multiplying out and then replacing all occurrences of x_1x_2 by the variable f.

For general n, there are 2^n points to consider (i.e., all choices of each variable at a bound), and the inequality descriptions in the space of $(f, x_1, \dots, x_n) \in \mathbb{R}^{n+1}$ get rather complicated (even for $n = 3$; see [17, 16]). It is frequent practice, both in modeling and software, to repeatedly use the McCormick inequalities when $n > 2$. Already the trilinear case, $n = 3$, is an interesting one for analysis. Here, we have three choices, which can be thought of as $f = (x_1x_2)x_3$, $f = (x_1x_3)x_2$ and $f = (x_2x_3)x_1$. Because the domain of each variable is its own interval $[a_i, b_i]$, the grouping can affect the quality of the convexification. In what follows, we analytically quantify the quality of these different convexification possibilities, in addition to the trilinear hull itself.

Our results are not just relevant to trilinear monomials in formulations. With the sBB approach for factorable formulations, our results are relevant whenever three quantities are multiplied. That is, as an expression DAG is created and auxiliary variables are introduced, a trilinear monomial will arise whenever three quantities (which can be complicated functions themselves) are multiplied.

In what follows, we use $(n+1)$-dimensional volume to compare different natural convexifications of graphs of functions of n variables on the box domain $[a_1, b_1] \times \cdots \times [a_n, b_n]$. We present a complete analytic analysis of the case of $n = 3$, for all choices of $0 \leq a_i < b_i$. It is perhaps surprising that this can be carried out, and probably less surprising that the analysis is quite complicated.

Volume as a measure for comparing relaxations was first proposed in [10]; also see [8] and [28]. In fact, the *practical* use of volume as a measure for comparing relaxations in the context of nonlinear mixed-integer optimization, foreshadowed by [10], was later validated computationally for a nonlinear version of the uncapactitated facility-location problem (see [9]). Specifically, using volume calculations, a main mathematical result of [10] is that weak formulations of facility-location problems are very close to strong formulations when the number of facilities is small compared to the number of customers. Then [9] showed that in this scenario, with a convex objective function, the weak formulation computationally out performs the strong formulation in the context of branch-and-bound. The emphasis in [10, 8, 28] was not on sBB nor on low-dimensional functions. Because those results pertained to varying dimension and related asymptotics, exactly how volumes are compared and scaled was important (in particular, see [10] which defines the "idealized radial distance"). Because we now focus on low-dimensional polytopes, the exact manner of comparison and scaling is much less relevant. Using volume as a measure corresponds to a uniform distribution of the optimal solution across a relaxation. This is justified in the context of *nonlinear* optimization if we want a measure that is robust across all formulations. One can well find situations where the volume measure is misleading. It would not make sense for evaluating polyhedral relaxations of the integer points in a polytope, if we were only concerned with *linear* objectives — in such a case, solutions are concentrated on the boundary and there are better measures available (see [10]). But if we are interested in a mathematically-tractable measure that robustly makes sense in the context of global optimization, volume is quite natural.

Motivated by two well-studied applications (the *Molecular Distance Geometry Problem* and the *Hartree-Fock Problem*), [4] first proposed volume in the context of sBB and monomials, but they leapfrogged to the case of $n = 4$ and took a mostly experimental approach. They demonstrated that there can be a significant difference in performance depending on grouping, and they offered some guidance based on computational experiments. At the time of that work, it appeared that developing precise formulae for volumes relevant to repeated McCormick was not tractable. With our present work on $n = 3$, it now seems possible that the case of $n = 4$ could be carried out (see §9

for an idea concerning how our results for $n=3$ could already be applied in practice to the $n=4$ case).

There has been considerable research on multilinear monomials and generalizations in the context of global optimization, notably [19, 14, 2, 21, 7, 18]. Our work adds to that literature.

In §2, we define the polytopes that we work with. In §3, we discuss the various alternatives for working with triple products. In §4, we present our main results and their consequences. In §§5–8, we present our proofs. In §9, we describe future directions for investigation. §10, an appendix, contains technical lemmas and calculations.

2. Double McCormick.

When using the double-McCormick technique to convexify trilinear monomials, a modeling/algorithmic choice is involved: we must choose to which pair of variables we apply the first iteration of McCormick. For the variables $x_i \in [a_i, b_i]$, $i = 1,2,3$, let $\mathcal{O}_i := a_i(b_j b_k) + b_i(a_j a_k)$. Then we can label the variables such that $\mathcal{O}_1 \le \mathcal{O}_2 \le \mathcal{O}_3$. In this manner, we can assume that

$$a_1 b_2 b_3 + b_1 a_2 a_3 \le b_1 a_2 b_3 + a_1 b_2 a_3 \le b_1 b_2 a_3 + a_1 a_2 b_3. \tag{Ω}$$

Given the trilinear monomial $f := x_1 x_2 x_3$, there are three choices of convexifications depending on the bilinear sub-monomial we convexify first. We could first group x_1 and x_2 and convexify $w = x_1 x_2$; after this, we are left with the monomial $f = w x_3$ which we can also convexify using McCormick. Alternatively, we could first group variables x_1 and x_3, or variables x_2 and x_3.

2.1. Convexification.

To see how to perform these convexifications in general, we show the double-McCormick convexification that first groups the variables x_i and x_j. Therefore we have $f = x_i x_j x_k$ and we let $w_{ij} = x_i x_j$ so $f = w_{ij} x_k$.

Convexifying $w_{ij} = x_i x_j$ we obtain the inequalities:

$$\begin{aligned}
w_{ij} - a_j x_i - a_i x_j + a_i a_j &\ge 0,\\
-w_{ij} + b_j x_i + a_i x_j - a_i b_j &\ge 0,\\
-w_{ij} + a_j x_i + b_i x_j - b_i a_j &\ge 0,\\
w_{ij} - b_j x_i - b_i x_j + b_i b_j &\ge 0.
\end{aligned}$$

Convexifying $f = w_{ij} x_k$ we obtain the inequalities:

$$\begin{aligned}
f - a_k w_{ij} - a_i a_j x_k + a_i a_j a_k &\ge 0,\\
-f + b_k w_{ij} + a_i a_j x_k - a_i a_j b_k &\ge 0,\\
-f + a_k w_{ij} + b_i b_j x_k - b_i b_j a_k &\ge 0,\\
f - b_k w_{ij} - b_i b_j x_k + b_i b_j b_k &\ge 0.
\end{aligned}$$

Using Fourier-Motzkin elimination, we then eliminate the variable w_{ij} to obtain the following system in our original variables f, x_i, x_j and x_k.

$$\begin{aligned}
&x_i - a_i && \ge 0, && (1)\\
&x_j - a_j && \ge 0, && (2)\\
&f - a_j a_k x_i - a_i a_k x_j - a_i a_j x_k + 2 a_i a_j a_k && \ge 0, && (3)\\
&f - a_j b_k x_i - a_i b_k x_j - b_i b_j x_k + a_i a_j b_k + b_i b_j b_k && \ge 0, && (4)\\
&- x_j + b_j && \ge 0, && (5)\\
&- x_i + b_i && \ge 0, && (6)\\
&f - b_j a_k x_i - b_i a_k x_j - a_i a_j x_k + a_i a_j a_k + b_i b_j a_k && \ge 0, && (7)\\
&f - b_j b_k x_i - b_i b_k x_j - b_i b_j x_k + 2 b_i b_j b_k && \ge 0, && (8)\\
&- f + b_j b_k x_i + a_i b_k x_j + a_i a_j x_k - a_i a_j b_k - a_i b_j b_k && \ge 0, && (9)\\
&- f + a_j b_k x_i + b_i b_k x_j + a_i a_j x_k - a_i a_j b_k - b_i a_j b_k && \ge 0, && (10)
\end{aligned}$$

$$
\begin{aligned}
&-x_k+b_k && \geq 0, && (11)\\
&-f+b_ja_kx_i+a_ia_kx_j+b_ib_jx_k-a_ib_ja_k-b_ib_ja_k && \geq 0, && (12)\\
&-f+a_ja_kx_i+b_ia_kx_j+b_ib_jx_k-b_ia_ja_k-b_ib_ja_k && \geq 0, && (13)\\
&x_k-a_k && \geq 0, && (14)\\
&f-a_ia_jx_k && \geq 0, && (15)\\
&-f+b_ib_jx_k && \geq 0. && (16)
\end{aligned}
$$

It is easy to see that the inequalities 15 and 16 are redundant: 15 is $a_ja_k(1)+a_ia_k(2)+(3)$, and 16 is $b_ja_k(6)+a_ia_k(5)+(12)$.

We use the following notation in what follows. For $i=1,2,3$, *system* i is defined to be the system of inequalities obtained by first grouping the pair of variables x_j and x_k, with j and k different from i. $\mathcal{P}_i$ is defined to be the solution set of this system.

2.2. Hull. As we noted earlier, a convex-hull representation for trilinear monomials is known. From [17], for any labeling that satisfies Ω (or even just the first inequality of Ω), this inequality system which we refer to as H is:

$$
\begin{aligned}
f-a_2a_3x_1-a_1a_3x_2-a_1a_2x_3+2a_1a_2a_3 &\geq 0, && (17)\\
f-b_2b_3x_1-b_1b_3x_2-b_1b_2x_3+2b_1b_2b_3 &\geq 0, && (18)\\
f-a_2b_3x_1-a_1b_3x_2-b_1a_2x_3+a_1a_2b_3+b_1a_2b_3 &\geq 0, && (19)\\
f-b_2a_3x_1-b_1a_3x_2-a_1b_2x_3+b_1b_2a_3+a_1b_2a_3 &\geq 0, && (20)\\
f-\tfrac{\eta_1}{b_1-a_1}x_1-b_1a_3x_2-b_1a_2x_3+\left(\tfrac{\eta_1a_1}{b_1-a_1}+b_1b_2a_3+b_1a_2b_3-a_1b_2b_3\right) &\geq 0, && (21)\\
f-\tfrac{\eta_2}{a_1-b_1}x_1-a_1b_3x_2-a_1b_2x_3+\left(\tfrac{\eta_2b_1}{a_1-b_1}+a_1a_2b_3+a_1b_2a_3-b_1a_2a_3\right) &\geq 0, && (22)\\
-f+a_2a_3x_1+b_1a_3x_2+b_1b_2x_3-b_1b_2a_3-b_1a_2a_3 &\geq 0, && (23)\\
-f+b_2a_3x_1+a_1a_3x_2+b_1b_2x_3-b_1b_2a_3-a_1b_2a_3 &\geq 0, && (24)\\
-f+a_2a_3x_1+b_1b_3x_2+b_1a_2x_3-b_1a_2b_3-b_1a_2a_3 &\geq 0, && (25)\\
-f+b_2b_3x_1+a_1a_3x_2+a_1b_2x_3-a_1b_2b_3-a_1b_2a_3 &\geq 0, && (26)\\
-f+a_2b_3x_1+b_1b_3x_2+a_1a_2x_3-b_1a_2b_3-a_1a_2b_3 &\geq 0, && (27)\\
-f+b_2b_3x_1+a_1b_3x_2+a_1a_2x_3-a_1b_2b_3-a_1a_2b_3 &\geq 0, && (28)\\
x_1-a_1 &\geq 0, && (29)\\
-x_1+b_1 &\geq 0, && (30)\\
x_2-a_2 &\geq 0, && (31)\\
-x_2+b_2 &\geq 0, && (32)\\
x_3-a_3 &\geq 0, && (33)\\
-x_3+b_3 &\geq 0, && (34)
\end{aligned}
$$

where $\eta_1=b_1b_2a_3-a_1b_2b_3-b_1a_2a_3+b_1a_2b_3$ and $\eta_2=a_1a_2b_3-b_1a_2a_3-a_1b_2b_3+a_1b_2a_3$.

We refer to the polytope defined as the feasible set of system H as $\mathcal{P}_H$. The extreme points of $\mathcal{P}_H$ are the 8 points that correspond to the $2^3=8$ choices of each x-variable at its upper or lower bound. We label these 8 points (all of the form $[f=x_1x_2x_3,x_1,x_2,x_3]$) as follows:

$$
v^1:=\begin{bmatrix}b_1a_2a_3\\b_1\\a_2\\a_3\end{bmatrix},\ v^2:=\begin{bmatrix}a_1a_2a_3\\a_1\\a_2\\a_3\end{bmatrix},\ v^3:=\begin{bmatrix}a_1a_2b_3\\a_1\\a_2\\b_3\end{bmatrix},\ v^4:=\begin{bmatrix}a_1b_2a_3\\a_1\\b_2\\a_3\end{bmatrix},
$$

$$
v^5:=\begin{bmatrix}a_1b_2b_3\\a_1\\b_2\\b_3\end{bmatrix},\ v^6:=\begin{bmatrix}b_1b_2b_3\\b_1\\b_2\\b_3\end{bmatrix},\ v^7:=\begin{bmatrix}b_1b_2a_3\\b_1\\b_2\\a_3\end{bmatrix},\ v^8:=\begin{bmatrix}b_1a_2b_3\\b_1\\a_2\\b_3\end{bmatrix}.
$$

Each alternative polyhedral convexification leads to a different system of inequalities (system i, $i=1,2,3$) and therefore a different polytope ($\mathcal{P}_i$, $i=1,2,3$) in $\mathbb{R}^4$ — all three contain the convex hull of the solution set of our original trilinear monomial (on the box domain), i.e. $\mathcal{P}_H$.

To establish if one of these three convexifications is better than another, we need to be able to compare these polytopes in a quantifiable manner. We take the (4-dimensional) volume as our measure, with the idea that a smaller volume corresponds to a tighter convexification.

For trilinear monomials with domain being a box (in the nonnegative orthant), we derive exact expressions for the (4-dimensional) volume for the convex hull of the set of solutions and also for each of the three possible double-McCormick convexifications. These volumes are in terms of six parameters (the upper and lower bounds on each of the three variables) and are rather complicated. By comparing the volume expressions, we are able to draw conclusions regarding the optimal way to perform double McCormick for trilinear monomials.

3. Alternatives. In practice, there are many possibilities for handling each product of three terms encountered in a formulation. A good choice, which may well be different for different triple products in the same formulation, ultimately depends on trading off the tightness of a relaxation with the overhead in working with it. For clarity, in the remainder of this section, we focus on different possible treatments of $f=x_1x_2x_3$.

One possibility is to use the full trilinear hull $\mathcal{P}_H$. This representation has the benefit of using no auxiliary variables. Another possibility to use the *convex-hull representation* (see [5], for example), writing $f=\sum_{j=1}^{8}\lambda_j v^j$, with $\sum_{j=1}^{8}\lambda_j=1$, $\lambda_j\geq 0$, for $j=1,2,\ldots,8$. This formulation has the drawback of utilizing *eight auxiliary variables*. But noticing that there are 5 linear equations, we can really reduce to *three auxiliary variables*. In fact, there is a very structured way to do this, where none of the λ_j variables are employed at all, and rather we introduce *three auxiliary variables* w_{12}, w_{13} and w_{23}, which represent the products x_1x_2, x_1x_3 and x_2x_3, respectively. A strong advantage of this last approach is when terms x_1x_2, x_1x_3 and x_2x_3 are also in the model under consideration. We wish to emphasize that projecting any of these convex-hull representations (reduced or not) down to the space of (f,x_1,x_2,x_3) yields again $\mathcal{P}_H$, and so all of these representations have the same bounding power.

We are advocating the *consideration* of double-McCormick relaxations as an alternative when warranted. We have identified the best among the double McCormicks and quantified the error in using it in preference to $\mathcal{P}_H$ (and, ipso facto, with any convex-hull or reduced convex hull representation). A double-McCormick relaxation involves only *one auxiliary variable* (and 8 inequalities). This can be particularly attractive when this particular auxiliary variable already appears in the model under consideration. Alternatively, especially when this particular auxiliary variable does *not* appear in the formulation, we can use the formulation with *zero auxiliary variables* (1-14). Recently (see [27]), we have computationally validated such an approach in the context of "box cubic programs"

$$\min_{x\in\mathbb{R}^n}\left\{\sum_{\{i,j,k\}} q_{ijk}\,x_i\,x_j\,x_k \;:\; x_i\in[a_i,b_i],\; i=1,2,\ldots,n\right\}.$$

In this type of problem, we can apply (1-14) *independently* for each trinomial, with no auxiliary variables at all, choosing the best double-McCormick for each trinomial, whenever the associated volume is close to the volume for $\mathcal{P}_H$. We have documented that this can happen quite a lot, and so it is a viable approach. It is important to emphasize that some of the negative experience with double McCormick is related to choosing the wrong one. Indeed, our mathematical and computational results indicate that there are many situations where: (i) the worst double McCormick is quite bad compared to the best one, and (ii) the best one is only slightly worse than $\mathcal{P}_H$ (and its convex-hull representations).

Besides any prescriptive use of double-McCormick relaxations, our results can simply be seen as quantifying the bounding advantage given by $\mathcal{P}_H$ and the various convex-hull representations (reduced or not) as compared to each of the possible double McCormick relaxations.

In some global-optimization software (e.g., `BARON` and `ANTIGONE`) the complicated inequality description of the trilinear hull is explicitly used. In other global-optimization software (e.g., `COUENNE` and `SCIP`) and as a technique at the formulation level, repeated McCormick is used for the trilinear case. It is by no means clear that either approach should be followed all of the time (though this currently seems to be the case), because of the solution-time tradeoff in using more complicated but stronger convexifications. This effect can be especially pronounced in the case of nonlinear optimization where solutions may not be on the boundary (see [9], for example). By quantifying the quality of different convexifications, we offer (i) firm and actionable means for deciding between them at run time and, (ii) some explanation for differing behavior of sBB software under different scenarios.

Finally, we note that the double-McCormick approach is often applied at the modeling level (see [11], for example). In particular, our results are highly relevant to modelers who simply use global-optimization software, often through a modeling language. An uniformed modeler can defeat clever software. In such a case, it is very useful for the user to know which double McCormick to use, because a bad one may negatively affect sBB performance, and all sBB software that we know will not capture *implicit* triple products in formulations.

4. Theorems. First we define the following twelve points in $\mathbb{R}^4$, where $j := i+1 \pmod 3$ and $k := i+2 \pmod 3$:

$$v_1^9 := \begin{bmatrix} \theta_1^1 \\ \theta_1^2 \\ a_2 \\ b_3 \end{bmatrix}, v_1^{10} := \begin{bmatrix} \theta_1^3 \\ \theta_1^4 \\ b_2 \\ a_3 \end{bmatrix}, v_1^{11} := \begin{bmatrix} \theta_1^5 \\ \theta_1^6 \\ b_2 \\ a_3 \end{bmatrix}, v_1^{12} := \begin{bmatrix} \theta_1^7 \\ \theta_1^8 \\ a_2 \\ b_3 \end{bmatrix}, v_2^9 := \begin{bmatrix} \theta_2^1 \\ b_1 \\ \theta_2^2 \\ a_3 \end{bmatrix}, v_2^{10} := \begin{bmatrix} \theta_2^3 \\ a_1 \\ \theta_2^4 \\ b_3 \end{bmatrix},$$

$$v_2^{11} := \begin{bmatrix} \theta_2^5 \\ a_1 \\ \theta_2^6 \\ b_3 \end{bmatrix}, v_2^{12} := \begin{bmatrix} \theta_2^7 \\ b_1 \\ \theta_2^8 \\ a_3 \end{bmatrix}, v_3^9 := \begin{bmatrix} \theta_3^3 \\ b_1 \\ a_2 \\ \theta_3^4 \end{bmatrix}, v_3^{10} := \begin{bmatrix} \theta_3^1 \\ a_1 \\ b_2 \\ \theta_3^2 \end{bmatrix}, v_3^{11} := \begin{bmatrix} \theta_3^7 \\ a_1 \\ b_2 \\ \theta_3^8 \end{bmatrix}, v_3^{12} := \begin{bmatrix} \theta_3^5 \\ b_1 \\ a_2 \\ \theta_3^6 \end{bmatrix},$$

where:

$$\theta_i^1 = a_i a_j a_k + \frac{a_j(b_k - a_k)(b_i b_j b_k - a_i a_j a_k)}{b_j b_k - a_j a_k}, \qquad \theta_i^2 = a_i + \frac{a_j(b_i - a_i)(b_k - a_k)}{b_j b_k - a_j a_k},$$

$$\theta_i^3 = a_i a_j a_k + \frac{a_k(b_j - a_j)(b_i b_j b_k - a_i a_j a_k)}{b_j b_k - a_j a_k}, \qquad \theta_i^4 = a_i + \frac{a_k(b_j - a_j)(b_i - a_i)}{b_j b_k - a_j a_k},$$

$$\theta_i^5 = \frac{b_j a_k(a_i b_j b_k - a_i a_j b_k - b_i a_j a_k + b_i a_j b_k)}{b_j b_k - a_j a_k}, \qquad \theta_i^6 = a_i + \frac{b_j(b_i - a_i)(b_k - a_k)}{b_j b_k - a_j a_k},$$

$$\theta_i^7 = \frac{a_j b_k(b_i b_j a_k - b_i a_j a_k - a_i b_j a_k + a_i b_j b_k)}{b_j b_k - a_j a_k}, \qquad \theta_i^8 = a_i + \frac{b_k(b_j - a_j)(b_i - a_i)}{b_j b_k - a_j a_k}.$$

Next, we state our main results.

THEOREM 4.1.

$$\begin{aligned}\mathrm{Vol}_{\mathcal{P}_H} = {} & (b_1 - a_1)(b_2 - a_2)(b_3 - a_3) \times \\ & \left(b_1(5b_2b_3 - a_2b_3 - b_2a_3 - 3a_2a_3) + a_1(5a_2a_3 - b_2a_3 - a_2b_3 - 3b_2b_3)\right)/24.\end{aligned}$$

THEOREM 4.2. *The set of extreme points of $\mathcal{P}_1$ is $\{v^1, \ldots, v^8\} \cup \{v_1^9, \ldots, v_1^{12}\}$. Moreover,*

$$\begin{aligned}\mathrm{Vol}_{\mathcal{P}_1} = {} & \mathrm{Vol}_{\mathcal{P}_H} + (b_1 - a_1)(b_2 - a_2)^2(b_3 - a_3)^2 \times \\ & \frac{3(b_1b_2a_3 - a_1b_2a_3 + b_1a_2b_3 - a_1a_2b_3) + 2(a_1b_2b_3 - b_1a_2a_3)}{24(b_2b_3 - a_2a_3)}.\end{aligned}$$

Theorem 4.3. *The set of extreme points of* $\mathcal{P}_2$ *is* $\{v^1,\ldots,v^8\}\cup\{v_2^9,\ldots,v_2^{12}\}$. *Moreover,*

$$\mathrm{Vol}_{\mathcal{P}_2} = \mathrm{Vol}_{\mathcal{P}_H} + \frac{(b_1-a_1)(b_2-a_2)^2(b_3-a_3)^2\left(5(a_1b_1b_3-a_1b_1a_3)+3(b_1^2a_3-a_1^2b_3)\right)}{24(b_1b_3-a_1a_3)}.$$

Theorem 4.4. *The set of extreme points of* $\mathcal{P}_3$ *is* $\{v^1,\ldots,v^8\}\cup\{v_3^9,\ldots,v_3^{12}\}$. *Moreover,*

$$\mathrm{Vol}_{\mathcal{P}_3} = \mathrm{Vol}_{\mathcal{P}_H} + \frac{(b_1-a_1)(b_2-a_2)^2(b_3-a_3)^2\left(5(a_1b_1b_2-a_1b_1a_2)+3(b_1^2a_2-a_1^2b_2)\right)}{24(b_1b_2-a_1a_2)}.$$

Our proofs in §§5–8 all assume that $a_1,a_2,a_3>0$. Next, we briefly explain why the theorems hold even when any of the a_i are zero. Taking the convex hull of a compact set is continuous (even 1-Lipschitz) in the Hausdorff metric (see [23, p. 51]). The volume functional is continuous (with respect to the Hausdorff metric) on the set K^n of convex bodies in $\mathbb{R}^n$ (see [24, Theorem 1.8.20; p. 68]). If two sets of m points in $\mathbb{R}^n$ are close as vectors in $\mathbb{R}^{mn}$, then they are also close in the Hausdorff metric. Therefore, the volume of the convex hull of a set of m points in $\mathbb{R}^n$ is a continuous function of the coordinates of the points. Also, the coordinates of the extreme points of our polytopes are all continuous functions (of the six parameters) at $a_i=0$. Finally, we note that the volume formulae that we derive are continuous functions (of the six parameters) at $a_i=0$. Therefore, those formulae are also correct when some $a_i=0$. We do note that we can also modify our constructions to handle these cases where some of the a_i are zero, but our continuity argument is much shorter.

Corollary 4.1. *For all values of the parameters* a_1,b_1,a_2,b_2,a_3,b_3, *meeting the conditions* (Ω), *we have:* $\mathrm{Vol}_{\mathcal{P}_H}\le\mathrm{Vol}_{\mathcal{P}_3}\le\mathrm{Vol}_{\mathcal{P}_2}\le\mathrm{Vol}_{\mathcal{P}_1}$.

From this we can see that with the variables ordered according to their upper and lower bounds per (Ω), the smallest volume will always be obtained by using system 3 (i.e., first grouping variables x_1 and x_2). In addition, for different values of the upper and lower bounds, we can precisely quantify the difference in volume of the alternative convexifications.

Moreover, by substituting $a_1=a_2=0$ and $b_1=b_2=1$ into the conditions (Ω), we can easily see the following corollary relevant to mixed-integer nonlinear optimization.

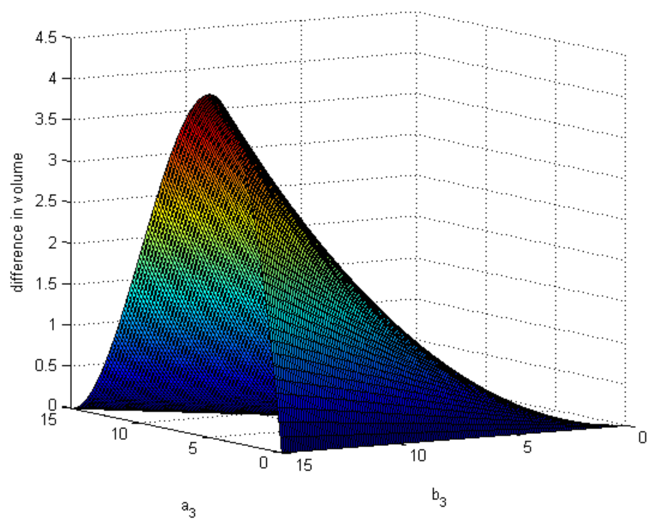
Corollary 4.2. *In the special case of two binary variables and one continuous variable, first grouping the two binary variables gives the convexification with the smallest volume.*

In this special case, we only have two parameters a_3 and b_3 and the volume formulae simplify considerably. In particular, for this special case, $\mathcal{P}_3$ is equivalent to $\mathcal{P}_H$, and $\mathcal{P}_1$ and $\mathcal{P}_2$ are equivalent. We compute the difference in volume between the two distinct choices of convexification and, in Figure 1, plot this expression as the parameters vary (satisfying $0\le a_3<b_3$). The following is easy to establish.

Corollary 4.3. *As* a_3 *and* b_3 *increase, the difference in volumes of* $\mathcal{P}_3$ *and* $\mathcal{P}_2$ *(or* $\mathcal{P}_1$*) becomes arbitrarily large. Additionally, for a fixed* b_3, *the greatest difference in volume occurs when* $a_3=b_3/3$.

Finally we note that in the special case in which $a_1=a_2=a_3=0$, each convexification reduces to the convex hull, which is a result of [21]. So in this case, *any* double-McCormick convexification has the power of the more-complicated inequality description of the convex hull. In fact, viewed this way, our results provide a quantified generalization of this result of [21]. We do wish to emphasize that because our results do not just apply to trilinear monomials on the formulation variables, but may well involve auxiliary variables, *the case of non-zero lower bounds is very relevant.*

FIGURE 1. Graph of difference in volume $\left(\frac{3a_3(b_3-a_3)^2}{24b_3}\right)$ vs. parameter values



5. Proof of Thm. 4.1. We compute the volume of $\mathcal{P}_H$ by constructing a triangulation. See Figure 2 for a diagram of the 8 extreme points of $\mathcal{P}_H$. Note that v^2, which has all of the variables at their lower bounds, is at the bottom of the "inner cube", and v^6, which has all of the variables at their upper bounds, is at the top of the "outer cube".

We use the fact that the volume of an n-simplex in $\mathbb{R}^n$ with vertices $(z^0, \ldots, z^n)$ is:

$$|\det(z^1 - z^0 \;\; z^2 - z^0 \;\; \ldots \;\; z^n - z^0)|/n!.$$

The volume of the 4-simplex with extreme points v^1, v^2, v^4, v^5 and v^6, which we define as $\mathcal{S} := \operatorname{conv}\{v^1, v^2, v^4, v^5, v^6\}$, is

$$(b_1 - a_1)^2(b_2 - a_2)(b_3 - a_3)(b_2b_3 - a_2a_3)/24.$$

A 4-simplex has 5 facets, each of which is a 3-simplex and is described by the hyperplane through a choice of 4 extreme points. To determine the facet-describing inequalities, we compute each hyperplane and then check the final point to obtain the direction of the inequality. The 5 facets of $\mathcal{S}$ are described as follows:

F^1 (hyperplane through points v^1, v^2, v^4, v^6):

$$-f + a_2a_3x_1 + a_1a_3x_2 + \frac{(a_1a_2a_3 - a_1b_2a_3 - b_1a_2a_3 + b_1b_2b_3)}{(b_3 - a_3)}x_3 - \frac{(a_1a_2a_3b_3 - a_1b_2a_3^2 - b_1a_2a_3^2 + b_1b_2a_3b_3)}{(b_3 - a_3)} \geq 0$$

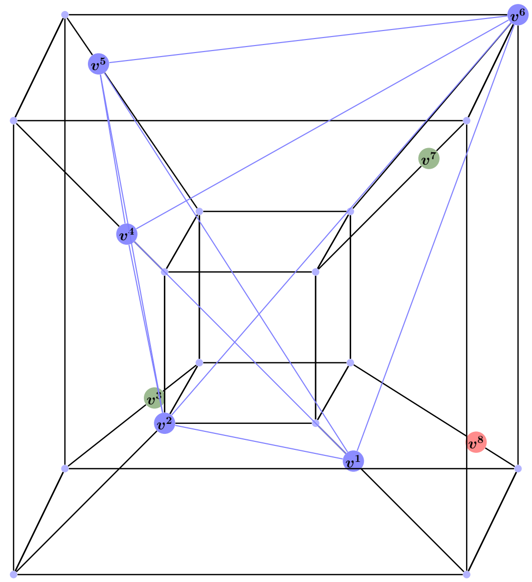
F^2 (hyperplane through points v^1, v^2, v^4, v^5):

$$f - a_2a_3x_1 - a_1a_3x_2 - a_1b_2x_3 + a_1a_2a_3 + a_1b_2a_3 \geq 0$$

F^3 (hyperplane through points v^1, v^2, v^5, v^6):

$$(b_3 - a_3)x_2 - (b_2 - a_2)x_3 + b_2a_3 - a_2b_3 \geq 0$$

FIGURE 2. Visual representation of simplex, $\mathcal{S}$, with extreme points v^1, v^2, v^4, v^5 and v^6, plus the other 3 convex hull extreme points v^3, v^7 and v^8.



F^4 (hyperplane through points v^1, v^4, v^5, v^6):

$$f - b_2b_3x_1 - \frac{(a_1b_2a_3 - a_1b_2b_3 - b_1a_2a_3 + b_1b_2b_3)}{(b_2 - a_2)}x_2 - a_1b_2x_3 + \frac{(-a_1a_2b_2b_3 + a_1b_2^2a_3 - b_1a_2b_2a_3 + b_1b_2^2b_3)}{(b_2 - a_2)} \geq 0$$

F^5 (hyperplane through points v^2, v^4, v^5, v^6):

$$-f + b_2b_3x_1 + a_1a_3x_2 + a_1b_2x_3 - a_1b_2a_3 - a_1b_2b_3 \geq 0$$

If a hyperplane H intersects a polytope P on a facet F, then H^+ (resp., H^-) denotes the half-space determined by H that contains (does not contain) P. If a point w is not in H but in H^+ (resp., H^-), then w is *beneath* (*beyond*) F (see [6, p. 78]).

We now compute the volume of $\text{conv}(\mathcal{S} \cup \{v^8\})$. To obtain the additional volume of this polytope compared with $\mathcal{S}$ we sum the volume of $\text{conv}(\{v^8\} \cup F)$ for each facet, F, of $\mathcal{S}$ such that v^8 is beyond that facet. To do this we first check each of the 5 facets to determine if v^8 is beneath or beyond that facet. To do this we substitute v^8 into the relevant inequality, and if the result is negative then v^8 lies beyond that facet.

It is easy to check that v^8 satisfies F^1 and F^5 and violates F^3. Using Lemma 10.1, we also check that v^8 satisfies both F^2 and F^4. From this we have that v^8 is beyond one facet, F^3. Therefore we need to calculate the volume of $\text{conv}(F^3 \cup \{v^8\}) = \text{conv}\{v^1, v^2, v^5, v^6, v^8\}$, this a a 4-simplex with volume:

$$(b_1 - a_1)^2(b_2 - a_2)(b_3 - a_3)(b_2b_3 - a_2a_3)/24.$$

We now have a new polytope which is $\text{conv}\{v^1, v^2, v^4, v^5, v^6, v^8\} = \text{conv}(\mathcal{S} \cup \{v^8\})$. We refer to this polytope as $\mathcal{Q}$. The volume of $\mathcal{Q}$ is given by the sum of the volumes of the two simplices we have computed thus far. The facets of $\mathcal{Q}$ are the facets of the original simplex without F^3, along with the facets of the 4-simplex: $\text{conv}(F^3 \cup \{v^8\})$ (again not including F^3 itself). A facet of $\text{conv}(F^3 \cup \{v^8\})$ is supported by a hyperplane through a choice of 4 of the 5 extreme points (points v^1, v^2, v^5, v^6 and v^8). As before, to determine these facet inequalities we compute each hyperplane

and then check the final point to obtain the direction of the inequality (note that we exclude the choice v^1, v^2, v^5, v^6 because this corresponds to F^3). The 4 facets are described below:

F^6 (plane through points v^1, v^2, v^5, v^8):

$$f - a_2a_3x_1 - \frac{(-a_1a_2a_3 + a_1b_2b_3 + b_1a_2a_3 - b_1a_2b_3)}{(b_2 - a_2)}x_2 - b_1a_2x_3 + \frac{(-a_1a_2^2a_3 + a_1a_2b_2b_3 - b_1a_2^2b_3 + b_1a_2b_2a_3)}{(b_2 - a_2)} \geq 0$$

F^7 (plane through points v^1, v^5, v^6, v^8):

$$f - b_2b_3x_1 - b_1b_3x_2 - b_1a_2x_3 + b_1a_2b_3 + b_1b_2b_3 \geq 0$$

F^8 (plane through points v^2, v^5, v^6, v^8):

$$-f + b_2b_3x_1 + b_1b_3x_2 + \frac{(-a_1a_2a_3 + a_1b_2b_3 + b_1a_2b_3 - b_1b_2b_3)}{(b_3 - a_3)}x_3 - \frac{(-a_1a_2a_3b_3 + a_1b_2b_3^2 + b_1a_2b_3^2 - b_1b_2a_3b_3)}{(b_3 - a_3)} \geq 0$$

F^9 (plane through points v^1, v^2, v^6, v^8):

$$-f + a_2a_3x_1 + b_1b_3x_2 + b_1a_2x_3 - b_1a_2a_3 - b_1a_2b_3 \geq 0$$

The facets of $\mathcal{Q} = \text{conv}\{v^1, v^2, v^4, v^5, v^6, v^8\}$ are therefore F^1, F^2, F^4, F^5, F^6, F^7, F^8 and F^9.

To obtain the entire volume of $\mathcal{P}_H$ we need to consider two further extreme points: v^3 and v^7. It would be convenient to add these points separately; i.e., compute the additional volume each produces when added to $\mathcal{Q}$, and sum the results. As the following lemma shows, this will give the correct volume if the intersection of the line segment between these points and $\mathcal{Q}$ is not empty.

Lemma 5.1. *Let P be a convex polytope and let w_1 and w_2 be points not in P. Let $L(w_1, w_2)$ be the line segment between w_1 and w_2. If $L(w_1, w_2) \cap P \neq \emptyset$, then $conv(P, w_1) \cup conv(P, w_2)$ is convex. Moreover, in this case, $conv(P, w_1, w_2) = conv(P, w_1) \cup conv(P, w_2)$.*

Proof. First we show that $conv(P, w_1) \cup conv(P, w_2)$ is convex. If we show that $L(w_1, w_2)$ is completely contained in $conv(P, w_1) \cup conv(P, w_2)$, then we will be done. Choose $z \in L(w_1, w_2) \cap P$. Now consider $L(w_1, z)$. Because $z \in P$, this whole line segment must be in $conv(P, w_1)$. Similarly consider $L(z, w_2)$; this whole line segment must be contained in $conv(P, w_2)$. Therefore the whole line segment $L(w_1, w_2)$ must be contained in $conv(P, w_1) \cup conv(P, w_2)$ and therefore this set is convex.

Next, we demonstrate that $conv(P, w_1, w_2) = conv(P, w_1) \cup conv(P, w_2)$. First, choose $y \in conv(P, w_1) \cup conv(P, w_2)$; therefore $y \in conv(P, w_1)$ or $y \in conv(P, w_2)$ (or both); in either case it is clear that $y \in conv(P, w_1, w_2)$. In the other direction, choose $y \in conv(P, w_1, w_2)$; therefore y can be written as a convex combination of the extreme points of P and w_1 and w_2. Because $conv(P, w_1) \cup conv(P, w_2)$ is convex, this means $y \in conv(P, w_1) \cup conv(P, w_2)$. Therefore the sets are equal as required. □

We refer to the midpoint of the line between w_1 and w_2 as $M(w_1, w_2)$. To show that the intersection of $L(v^3, v^7)$ and $\mathcal{Q}$ is non-empty, consider the midpoint

$$M(v^3, v^7) = \left[\ \tfrac{a_1a_2b_3 + b_1b_2a_3}{2}\ \ \tfrac{a_1+b_1}{2}\ \ \tfrac{a_2+b_2}{2}\ \ \tfrac{b_3+a_3}{2}\ \right].$$

We show that this point satisfies each of the inequalities of $\mathcal{Q}$ by substituting into each inequality and checking the result. By showing that each resulting quantity is nonnegative, we conclude that the midpoint intersects $\mathcal{Q}$. It is easy to see that the midpoint $M(v^3, v^7)$ satisfies F^1, F^5, F^8 and F^9. Using Lemma 10.1, we also check that $M(v^3, v^7)$ satisfies F^2, F^4, F^6 and F^7. Therefore $\text{conv}(\mathcal{Q} \cup \{v^3\}) \cup \text{conv}(\mathcal{Q} \cup \{v^7\}) = \text{conv}(\mathcal{Q} \cup \{v^3\} \cup \{v^7\}) = \mathcal{P}_H$.

5.1. Computing the (additional) volume of conv($\mathcal{Q} \cup \{v^3\}$). We now compute the additional volume of conv($\mathcal{Q} \cup \{v^3\}$) compared to the volume of $\mathcal{Q}$. To obtain this, we sum the volumes of conv($\{v^3\} \cup F$) for each facet, F, of $\mathcal{Q}$ such that v^3 is beyond that facet. We substitute v^3 into each relevant inequality, and if the result is negative then v^3 lies beyond that facet. It is easy to see that v^3 satisfies F^5, F^7 and F^9 and violates F^2, F^6 and F^8. It can then be checked that v^3 satisfies F^1 using Lemma 10.4 (with $A = b_2, B = a_2, C = (b_1b_3 - a_1a_3), D = (2a_1a_3 - a_1b_3 - b_1a_3)$). We also check that v^3 satisfies F^4 using Lemma 10.4 (with $A = b_2, B = a_2, C = (a_1a_3 - 2a_1b_3 + b_1b_3), D = (a_1b_3 - b_1a_3)$) and Lemma 10.1.

From this we know that v^3 is beyond F^2, F^6 and F^8; therefore we need to compute the volume of the convex hulls of v^3 with each of these facets.

The polytope conv($F^2 \cup \{v^3\}$) = conv$\{v^1, v^2, v^4, v^5, v^3\}$ is a 4-simplex with volume:

$$a_1(b_1 - a_1)(b_2 - a_2)^2(b_3 - a_3)^2/24.$$

The polytope conv($F^6 \cup \{v^3\}$) = conv$\{v^1, v^2, v^5, v^8, v^3\}$ is a 4-simplex with volume:

$$a_2(b_1 - a_1)^2(b_2 - a_2)(b_3 - a_3)^2/24.$$

The polytope conv($F^8 \cup \{v^3\}$) = conv$\{v^2, v^5, v^6, v^8, v^3\}$ is a 4-simplex with volume:

$$b_3(b_1 - a_1)^2(b_2 - a_2)^2(b_3 - a_3)/24.$$

5.2. Computing the (additional) volume of conv($\mathcal{Q} \cup \{v^7\}$). We now compute the additional volume of conv($\mathcal{Q} \cup \{v^7\}$) compared to the volume of $\mathcal{Q}$. To obtain this, we sum the volumes of conv($\{v^7\} \cup F$) for each facet, F, of $\mathcal{Q}$ such that v^7 is beyond that facet. We substitute v^7 into each relevant inequality, and if the result is negative then v^7 lies beyond that facet. It is easy to see that v^7 satisfies F^2, F^5 and F^9 and violates F^1, F^4 and F^7. It can then be checked that v^7 satisfies F^6 using Lemma 10.4 (with $A = b_2, B = a_2, C = (b_1a_3 - a_1b_3), D = (a_1a_3 - 2b_1a_3 + b_1b_3)$) and Lemma 10.1. We also check that v^7 satisfies F^8 using Lemma 10.4 (with $A = b_2, B = a_2, C = (2b_1b_3 - a_1b_3 - b_1a_3), D = (a_1a_3 - b_1b_3)$).

From this we know that v^7 is beyond F^1, F^4 and F^7, therefore we need to compute the volume of the convex hulls of v^7 with each of these facets.

The polytope conv($F^1 \cup \{v^7\}$) = conv$\{v^1, v^2, v^4, v^6, v^7\}$ is a 4-simplex with volume:

$$a_3(b_1 - a_1)^2(b_2 - a_2)^2(b_3 - a_3)/24.$$

The polytope conv($F^4 \cup \{v^7\}$) = conv$\{v^1, v^4, v^5, v^6, v^7\}$ is a 4-simplex with volume:

$$b_2(b_1 - a_1)^2(b_2 - a_2)(b_3 - a_3)^2/24.$$

The polytope conv($F^7 \cup \{v^7\}$) = conv$\{v^1, v^5, v^6, v^8, v^7\}$ is a 4-simplex with volume:

$$b_1(b_1 - a_1)(b_2 - a_2)^2(b_3 - a_3)^2/24.$$

To compute the volume of $\mathcal{P}_H$, we sum the volume of the appropriate eight simplices, and we obtain the volume of $\mathcal{P}_H$ as stated in Theorem 4.1. □

Table 1

Ineq	$M(v_3^9, v_3^{10})$	$M(v_3^9, v_3^{11})$	$M(v_3^9, v_3^{12})$	$M(v_3^{10}, v_3^{11})$	$M(v_3^{10}, v_3^{12})$
17	immediate	immediate	immediate	immediate	immediate
18	immediate	immediate	immediate	immediate	immediate
19	immediate	by Lemma 10.1	immediate	by Lemma 10.1	by Lemma 10.1
20	immediate	by Lemma 10.1	by Lemma 10.1	immediate	by Lemma 10.1
21	immediate	by Lemma 10.1	immediate	by Lemma 10.1	by Lemma 10.1
22	immediate	by Lemma 10.1	by Lemma 10.1	immediate	by Lemma 10.1
23	immediate	immediate	immediate	immediate	immediate
24	immediate	immediate	immediate	immediate	immediate
25	see 35	see §10.2	immediate	see 36	see 37
26	see 38	see 39	see 36	immediate	See §10.3
27	immediate	immediate	immediate	immediate	immediate
28	immediate	immediate	immediate	immediate	immediate
29	immediate	immediate	immediate	immediate	immediate
30	immediate	immediate	immediate	immediate	immediate
31	immediate	immediate	immediate	immediate	immediate
32	immediate	immediate	immediate	immediate	immediate
33	immediate	immediate	immediate	immediate	immediate
34	immediate	immediate	immediate	immediate	immediate

6. Proof of Thm. 4.4.

We compute the volume of the convex hull of the 12 extreme points which we claim are exactly the extreme points of system 3. In computing the volume of this polytope, we also prove that these are the correct extreme points and that we have therefore computed the volume of $\mathcal{P}_3$.

The relevant points are the eight extreme points of $\mathcal{P}_H$, plus an additional four points. Because we have already computed the volume of $\mathcal{P}_H$, to compute the volume of $\mathcal{P}_3$, we need to compute the additional volume, compared with $\mathcal{P}_H$, added by these four extra extreme points. To show that this is indeed the volume of $\mathcal{P}_3$, we keep track of which facets need to be deleted and added to the system of inequalities as we go. When this is complete, we have exactly system 3, and therefore we must also have the correct extreme points.

We begin with system H from §2.2. As discussed in §5, it would be convenient to add the four points to $\mathcal{P}_H$ separately; i.e., compute the additional volume each produces when added to $\mathcal{P}_H$, and sum the results. To show that we can add two points separately and obtain the correct volume, we show that the intersection of the line segment between these points and $\mathcal{P}_H$ is non-empty (Lemma 5.1).

We show that we can add v_3^9 separately, v_3^{10} separately, and then v_3^{11} and v_3^{12} together by considering the midpoints of the line segments between the relevant points. We consider $L(v_3^9, v_3^{10})$, $L(v_3^9, v_3^{11})$, $L(v_3^9, v_3^{12})$, $L(v_3^{10}, v_3^{11})$ and $L(v_3^{10}, v_3^{12})$. We show that the midpoint of each line segment satisfies each of the inequalities of $\mathcal{P}_H$ by substituting this point into each inequality and checking the result. See Table 1 for a summary of the resulting substitutions. The table notes whether non-negativity of the resulting quantity follows immediately (after factoring), or by use of a technical lemma (after further explanation in the appendix), or after being rewritten in the way referenced in Figure 3. Because we have shown that each resulting quantity is nonnegative, we conclude that each of the midpoints intersect $\mathcal{P}_H$, and therefore we can add v_3^9 separately, v_3^{10} separately, and then v_3^{11} and v_3^{12} together.

6.1. Computing the (additional) volume of conv($\mathcal{P}_H \cup \{v_3^9\}$).

We now compute the additional volume of $\text{conv}(\mathcal{P}_H \cup \{v_3^9\})$ compared to the volume of $\mathcal{P}_H$. To do this we sum the volumes of $\text{conv}(\{v_3^9\} \cup F)$ for each facet, F, of $\mathcal{P}_H$ such that v_3^9 is beyond that facet. We substitute v_3^9 into each inequality of system H, and we immediately see that it satisfies every inequality except 25.

Figure 3. For Table 1

$$\frac{(b_2-a_2)(b_1-a_1)\left(b_1b_3(b_2-a_2)+a_2a_3(b_1-a_1)\right)}{2(b_1b_2-a_1a_2)} \tag{35}$$

$$\frac{(b_2b_3-a_2a_3)(b_1-a_1)+(b_1b_3-a_1a_3)(b_2-a_2)}{2} \tag{36}$$

$$\frac{(b_2-a_2)(b_1-a_1)\left(b_1(b_2b_3-a_2a_3)+a_2(b_1b_3-a_1a_3)\right)}{2(b_1b_2-a_1a_2)} \tag{37}$$

$$\frac{(b_2-a_2)(b_1-a_1)(b_2b_3(b_1-a_1)+a_1a_3(b_2-a_2))}{2(b_1b_2-a_1a_2)} \tag{38}$$

$$\frac{(b_2-a_2)(b_1-a_1)\left(b_2(b_1b_3-a_1a_3)+a_1(b_2b_3-a_2a_3)\right)}{2(b_1b_2-a_1a_2)} \tag{39}$$

From this we know that v_3^9 is beyond only one facet. The extreme points that lie on this facet are points v^1, v^2, v^6 and v^8. The polytope $\text{conv}\{v^1, v^2, v^6, v^8, v_3^9\}$ is a 4-simplex with volume:

$$b_1a_2(b_1-a_1)^2(b_2-a_2)^2(b_3-a_3)^2/(24(b_1b_2-a_1a_2))\,.$$

The facets of $\text{conv}(\mathcal{P}_H \cup \{v_3^9\})$ are the facets of $\mathcal{P}_H$ except inequality 25. We see this by computing the four additional facets that come from adding v_3^9 and noting they are already contained in system H:

- The facet through points v^1, v^2, v^6 and v_3^9 is 23.
- The facet through points v^1, v^2, v^8 and v_3^9 is 31.
- The facet through points v^1, v^6, v^8 and v_3^9 is 30.
- The facet through points v^2, v^6, v^8 and v_3^9 is 27.

6.2. Computing the (additional) volume of $\text{conv}(\mathcal{P}_H \cup \{v_3^{10}\})$. We now compute the additional volume of $\text{conv}(\mathcal{P}_H \cup \{v_3^{10}\})$ compared to the volume of $\mathcal{P}_H$. To do this we sum the volumes of $\text{conv}(\{v_3^{10}\} \cup F)$ for each facet, F, of $\mathcal{P}_H$ such that v_3^{10} is beyond that facet. We substitute v_3^{10} into each inequality of system H, and we immediately see that every inequality is satisfied except 26.

From this we know that v_3^{10} is beyond only one facet. The extreme points that lie on this facet are points v^2, v^4, v^5 and v^6. The polytope $\text{conv}\{v^2, v^4, v^5, v^6, v_3^{10}\}$ is a 4-simplex with volume:

$$a_1b_2(b_1-a_1)^2(b_2-a_2)^2(b_3-a_3)^2/(24(b_1b_2-a_1a_2))\,.$$

The facets of $\text{conv}(\mathcal{P}_H \cup \{v_3^{10}\})$ are the facets of $\mathcal{P}_H$ except inequality 26. We see this by computing the four additional facets that come from adding v_3^{10} and noting that they are already contained in system H:

- The facet through points v^2, v^4, v^5 and v_3^{10} is 29.
- The facet through points v^2, v^4, v^6 and v_3^{10} is 24.
- The facet through points v^2, v^5, v^6 and v_3^{10} is 28.
- The facet through points v^4, v^5, v^6 and v_3^{10} is 32.

6.3. Computing the (additional) volume of $\text{conv}(\mathcal{P}_H \cup \{v_3^{11}\} \cup \{v_3^{12}\})$. We now compute the additional volume of $\text{conv}(\mathcal{P}_H \cup \{v_3^{11}\} \cup \{v_3^{12}\})$ compared to the volume of $\mathcal{P}_H$. Because $L(v_3^{11}, v_3^{12})$ lies entirely outside of $\mathcal{P}_3$, we need to add them sequentially.

We first compute the additional volume of $\text{conv}(\mathcal{P}_H \cup \{v_3^{11}\})$ compared to the volume of $\mathcal{P}_H$. As we have done previously, we sum the volumes of $\text{conv}(\{v_3^{11}\} \cup F)$ for each facet, F, of $\mathcal{P}_H$ such that v_3^{11} is beyond that facet. We substitute v_3^{11} into each relevant inequality, and if the result is

negative then v_3^{11} lies beyond that facet. It is immediate that v_3^{11} violates inequalities 19–22 and satisfies inequalities 17-18, 23-24 and 26-34. To see that inequality 25 is also satisfied see §10.4.

Therefore we have that v_3^{11} is beyond four facets, and we need to compute the volume of the convex hulls of v_3^{11} with each of these facets.

The extreme points that lie on the first facet are points v^1, v^3, v^5 and v^8. The polytope $\text{conv}\{v^1, v^3, v^5, v^8, v_3^{11}\}$ is a 4-simplex with volume:

$$a_1b_1(b_1-a_1)(b_2-a_2)^3(b_3-a_3)^2/(24(b_1b_2-a_1a_2))\,.$$

The extreme points that lie on the second facet are points v^1, v^4, v^5 and v^7. The polytope $\text{conv}\{v^1, v^4, v^5, v^7, v_3^{11}\}$ is a 4-simplex with volume:

$$a_1b_2(b_1-a_1)^2(b_2-a_2)^2(b_3-a_3)^2/(24(b_1b_2-a_1a_2))\,.$$

The extreme points that lie on the third facet are points v^1, v^5, v^7 and v^8. The polytope $\text{conv}\{v^1, v^5, v^7, v^8, v_3^{11}\}$ is a 4-simplex with volume:

$$a_1b_1(b_1-a_1)(b_2-a_2)^3(b_3-a_3)^2/(24(b_1b_2-a_1a_2))\,.$$

The extreme points that lie on the fourth facet are points v^1, v^3, v^4 and v^5. The polytope $\text{conv}\{v^1, v^3, v^4, v^5, v_3^{11}\}$ is a 4-simplex with volume:

$$a_1b_2(b_1-a_1)^2(b_2-a_2)^2(b_3-a_3)^2/(24(b_1b_2-a_1a_2))\,.$$

We now have a new polytope which is $\text{conv}(\mathcal{P}_H \cup \{v_3^{11}\})$. We refer to this polytope as $\mathcal{T}_3$, and we compute the facets of $\mathcal{T}_3$.

We begin with the facets of $\mathcal{P}_H$ and delete the four facets that v_3^{11} violated (19-22). Let us call this system $\mathcal{T}_3^-$. Now consider the four simplices we dealt with when computing the additional volume produced with v_3^{11}. Each of these simplices has 5 facets; one of which corresponds to a deleted facet of $\mathcal{P}_H$.

The remaining 4 facets of the first simplex are described by the planes through the following sets of points: $\{v^1, v^3, v^5, v_3^{11}\}$, $\{v^1, v^3, v^8, v_3^{11}\}$, $\{v^1, v^5, v^8, v_3^{11}\}$ and $\{v^3, v^5, v^8, v_3^{11}\}$.

The remaining 4 facets of the second simplex are described by the planes through the following sets of points: $\{v^1, v^4, v^5, v_3^{11}\}$, $\{v^1, v^4, v^7, v_3^{11}\}$, $\{v^1, v^5, v^7, v_3^{11}\}$ and $\{v^4, v^5, v^7, v_3^{11}\}$.

The remaining 4 facets of the third simplex are described by the planes through the following sets of points: $\{v^1, v^5, v^7, v_3^{11}\}$, $\{v^1, v^5, v^8, v_3^{11}\}$, $\{v^1, v^7, v^8, v_3^{11}\}$ and $\{v^5, v^7, v^8, v_3^{11}\}$.

The remaining 4 facets of the fourth simplex are described by the planes through the following sets of points: $\{v^1, v^3, v^4, v_3^{11}\}$, $\{v^1, v^3, v^5, v_3^{11}\}$, $\{v^1, v^4, v^5, v_3^{11}\}$ and $\{v^3, v^4, v^5, v_3^{11}\}$.

Consider these sixteen facets and exclude the facets that are shared by more than one simplex. This leaves eight facets.

We can compute these eight facets to obtain the following:

- The facet through points v^1, v^3, v^8 and v_3^{11} is

$$\begin{aligned}
&\frac{1}{b_1b_2-a_1a_2}\big(-a_1^2a_2^2b_3 + a_1^2a_2b_3x_2 - a_1b_1a_2^2a_3 + a_1b_1a_2^2x_3 \\
&\quad + a_1a_2^2b_3x_1 + a_1b_1a_2b_2a_3 + a_1b_1a_2a_3x_2 - a_1b_1a_2b_3x_2 - a_1b_1b_2a_3x_2 \\
&\quad + b_1^2a_2b_2b_3 - b_1^2a_2b_2x_3 - b_1a_2b_2b_3x_1 - a_1a_2f + b_1b_2f\big) \geq 0.
\end{aligned} \tag{40}$$

- The facet through points v^3, v^5, v^8 and v_3^{11} is

$$f - a_2b_3x_1 - a_1b_3x_2 - b_1b_2x_3 + a_1a_2b_3 + b_1b_2b_3 \geq 0. \tag{41}$$

- The facet through points v^1, v^4, v^7 and v_3^{11} is

$$f - b_2a_3x_1 - b_1a_3x_2 - a_1a_2x_3 + a_1a_2a_3 + b_1b_2a_3 \geq 0. \tag{42}$$

- The facet through points v^4, v^5, v^7 and v_3^{11} is 32.
- The facet through points v^1, v^7, v^8 and v_3^{11} is

$$\begin{aligned} &\frac{1}{b_1b_2 - a_1a_2}\big(-a_1b_1a_2^2a_3 + a_1b_1a_2^2x_3 + a_1b_1a_2a_3x_2 - a_1b_1a_2b_2b_3 + a_1a_2b_2b_3x_1 + b_1a_2b_2a_3x_1 \\ &\quad + b_1^2a_2b_2b_3 - b_1^2a_2b_2x_3 - b_1a_2b_2b_3x_1 + b_1^2b_2^2a_3 - b_1^2b_2a_3x_2 - b_1b_2^2a_3x_1 - a_1a_2f + b_1b_2f\big) \geq 0. \end{aligned} \tag{43}$$

- The facet through points v^5, v^7, v^8 and v_3^{11} is 18.
- The facet through points v^1, v^3, v^4 and v_3^{11} is 17.
- The facet through points v^3, v^4, v^5 and v_3^{11} is 29.

There are four inequalities that are not already contained in system $\mathcal{T}_3^-$, we add these and in doing so obtain the system of inequalities that describes $\mathcal{T}_3 = \text{conv}(\mathcal{P}_H \cup \{v_3^{11}\})$.

We now compute the additional volume of $\text{conv}(\mathcal{T}_3 \cup \{v_3^{12}\})$ compared to the volume of $\mathcal{T}_3$. As we have done previously, we sum the volumes of $\text{conv}(\{v_3^{12}\} \cup F)$ for each facet, F, of $\mathcal{T}_3$ such that v_3^{12} is beyond that facet. We substitute v_3^{12} into each relevant inequality (i.e., the system of inequalities that describes $\mathcal{T}_3$) and if the result is negative then v_3^{12} lies beyond that facet. It is immediately clear that v_3^{12} satisfies inequalities 17-18, 23-25, 27-34 and 41-42. We can also see immediately that v_3^{12} violates inequalities 40 and 43. To see that inequality 26 is also satisfied see §10.4.

Therefore we see that v_3^{12} is beyond two facets and we need to compute the volume of the convex hull of v_3^{12} with each of these facets.

The extreme points that lie on the first facet are points v^1, v^3, v^8 and v_3^{11}. The polytope $\text{conv}\{v^1, v^3, v^8, v_3^{11}, v_3^{12}\}$ is a 4-simplex with volume:

$$b_1a_2(b_1 - a_1)^2(b_2 - a_2)^2(b_3 - a_3)^2/(24(b_1b_2 - a_1a_2))\,.$$

The extreme points that lie on the second facet are points v^1, v^7, v^8 and v_3^{11}. The polytope $\text{conv}\{v^1, v^7, v^8, v_3^{11}, v_3^{12}\}$ is a 4-simplex with volume:

$$b_1a_2(b_1 - a_1)^2(b_2 - a_2)^2(b_3 - a_3)^2/(24(b_1b_2 - a_1a_2))\,.$$

We now compute the additional facets; we take the four facets from adding each simplex and delete the facet that repeats. This leaves us with the following six facets to compute:

- The facet through points v^1, v^7, v^8 and v_3^{12} is 30.
- The facet through points v^1, v^7, v_3^{11} and v_3^{12} is 42.
- The facet through points v^7, v^8, v_3^{11} and v_3^{12} is 18.
- The facet through points v^1, v^3, v^8 and v_3^{12} is 31.
- The facet through points v^1, v^3, v_3^{11} and v_3^{12} is 17.
- The facet through points v^3, v^8, v_3^{11} and v_3^{12} is 41.

By adding and deleting the appropriate facets to and from system H, we see that we arrive at system 3.

Therefore, to compute the volume of $\mathcal{P}_3$, we sum the volume of $\mathcal{P}_H$ with that of the appropriate eight simplices, and we obtain our result. $\square$

Table 2

Ineq	$M(v_1^9, v_1^{10})$	$M(v_1^9, v_1^{11})$	$M(v_1^9, v_1^{12})$	$M(v_1^{10}, v_1^{11})$	$M(v_1^{10}, v_1^{12})$
17	immediate	immediate	immediate	immediate	immediate
18	immediate	immediate	immediate	immediate	immediate
19	see §10.5	see 44 and Lemma 10.1	immediate	see 45 and Lemma 10.1	see §10.11
20	see §10.6	see §10.7	see 45 and Lemma 10.1	immediate	see 46 and Lemma 10.1
21	see §10.17	see §10.8	by Lemma 10.1	by Lemma 10.1	see §10.12
22	see §10.18	see §10.9	by Lemma 10.1	by Lemma 10.1	see §10.13
23	immediate	immediate	immediate	immediate	immediate
24	see 47	see 48	see 49	immediate	see §10.14
25	immediate	immediate	immediate	immediate	immediate
26	immediate	immediate	immediate	immediate	immediate
27	see 50	See §10.10	immediate	see 49	see 51
28	immediate	immediate	immediate	immediate	immediate
29	immediate	immediate	immediate	immediate	immediate
30	immediate	immediate	immediate	immediate	immediate
31	immediate	immediate	immediate	immediate	immediate
32	immediate	immediate	immediate	immediate	immediate
33	immediate	immediate	immediate	immediate	immediate
34	immediate	immediate	immediate	immediate	immediate

7. Proof of Thm. 4.2. As with Theorem 4.4, we compute the volume of the convex hull of the 12 extreme points which we claim are exactly the extreme points of system 1. In computing the volume of this polytope, we also prove that these are the correct extreme points and that we have therefore computed the volume of $\mathcal{P}_1$.

The relevant points are the eight extreme points of $\mathcal{P}_H$, plus an additional four points. Because we have already computed the volume of $\mathcal{P}_H$, to compute the volume of $\mathcal{P}_1$ we need to compute the additional volume, compared with $\mathcal{P}_H$, added by these four extra extreme points. To show that this is indeed the volume of $\mathcal{P}_1$, we keep track of which facets need to be deleted and added to the system of inequalities as we go. When this is complete, we have exactly system 1 and therefore we must also have the correct extreme points.

We begin with system H which can be found in §2.2, and we use the same principles as we used in the previous proof to compute the volume of $\mathcal{P}_3$.

First we argue that we can add v_1^9 to $\mathcal{P}_H$ separately, v_1^{10} to $\mathcal{P}_H$ separately and then v_1^{11} and v_1^{12} together. To do this, we show that the midpoint of the line segment between v_1^9 and all other additional points (v_1^{10}, v_1^{11} and v_1^{12}) intersects $\mathcal{P}_H$. We also show this is true for v_1^{10}.

As in the previous proof we refer to the midpoint of the line between v_i^j and v_i^k as $M(v_i^j, v_i^k)$ and we show that the midpoint of each line satisfies each of the inequalities of $\mathcal{P}_H$ by substituting this point into each inequality and checking the result. See Table 2 for a summary of the resulting substitutions. The table notes whether nonnegativity of the resulting quantity follows immediately (after factoring), or by using a technical lemma, after further explanation in the appendix or after being rewritten in the way referenced in Figure 4. Because we have shown that each resulting quantity is nonnegative, we know that the midpoint intersects $\mathcal{P}_H$, and therefore we can add v_1^9 separately, v_1^{10} separately, and then v_1^{11} and v_1^{12} together.

7.1. Computing the (additional) volume of conv($\mathcal{P}_H \cup \{v_1^9\}$). We now compute the additional volume of conv($\mathcal{P}_H \cup \{v_1^9\}$) compared to the volume of $\mathcal{P}_H$. To do this, we sum the volumes of conv($\{v_1^9\} \cup F$) for each facet, F, of $\mathcal{P}_H$ such that v_1^9 is beyond that facet. We substitute v_1^9 into the 18 relevant inequalities (17-34) and immediately see that it satisfies 17-19, 23-26 and 28-34. It is also immediate to see that inequality 27 is violated. To show that the remaining three inequalities are satisfied (20, 21 and 22) see §10.15, §10.19 and §10.20.

FIGURE 4. For Table 2

$$\frac{(b_3-a_3)(b_2-a_2)\left(b_3(b_1a_2-a_1b_2)+a_2(b_1a_3-a_1b_3)\right)}{2(b_2b_3-a_2a_3)} \tag{44}$$

$$\frac{(b_1a_3-a_1b_3)(b_2-a_2)+(b_1a_2-a_1b_2)(b_3-a_3)}{2} \tag{45}$$

$$\frac{(b_3-a_3)(b_2-a_2)\left(b_2(b_1a_3-a_1b_3)+a_3(b_1a_2-a_1b_2)\right)}{2(b_2b_3-a_2a_3)} \tag{46}$$

$$\frac{(b_3-a_3)(b_2-a_2)\left(b_1b_2(b_3-a_3)+a_1a_3(b_2-a_2)\right)}{2(b_2b_3-a_2a_3)} \tag{47}$$

$$\frac{(b_3-a_3)(b_2-a_2)\left(b_2(b_1b_3-a_1a_3)+a_3(b_1b_2-a_1a_2)\right)}{2(b_2b_3-a_2a_3)} \tag{48}$$

$$\frac{(b_1b_3-a_1a_3)(b_2-a_2)+(b_1b_2-a_1a_2)(b_3-a_3)}{2} \tag{49}$$

$$\frac{(b_3-a_3)(b_2-a_2)\left(b_1b_3(b_2-a_2)+a_1a_2(b_3-a_3)\right)}{2(b_2b_3-a_2a_3)} \tag{50}$$

$$\frac{(b_3-a_3)(b_2-a_2)\left(b_3(b_1b_2-a_1a_2)+a_2(b_1b_3-a_1a_3)\right)}{2(b_2b_3-a_2a_3)} \tag{51}$$

From this we know that v_1^9 is beyond only one facet. The extreme points that lie on this facet are points v^2, v^3, v^6 and v^8. The polytope $\text{conv}\{v^2, v^3, v^6, v^8, v_1^9\}$ is a 4-simplex with volume:

$$a_2b_3(b_1-a_1)^2(b_2-a_2)^2(b_3-a_3)^2/(24(b_2b_3-a_2a_3))\,.$$

The facets of $\text{conv}(\mathcal{P}_H \cup \{v_1^9\})$ are the facets of $\mathcal{P}_H$ except inequality 27. We see this by computing the four additional facets that come from adding v_1^9 and noting that they are already contained in system H:

- The facet through points v^2, v^3, v^6 and v_1^9 is 28.
- The facet through points v^2, v^3, v^8 and v_1^9 is 31.
- The facet through points v^2, v^6, v^8 and v_1^9 is 25.
- The facet through points v^3, v^6, v^8 and v_1^9 is 34.

7.2. Computing the (additional) volume of conv($\mathcal{P}_H \cup \{v_1^{10}\}$). We now compute the additional volume of $\text{conv}(\mathcal{P}_H \cup \{v_1^{10}\})$ compared to the volume of $\mathcal{P}_H$. To do this, we sum the volumes of $\text{conv}(\{v_1^{10}\} \cup F)$ for each facet, F, of $\mathcal{P}_H$ such that v_1^{10} is beyond that facet. We substitute v_1^{10} into the 18 relevant inequalities and immediately see that it satisfies 17, 18, 20, 23 and 25-34. It is also immediate to see that inequality 24 is violated. To show that the remaining three inequalities are satisfied (19, 21 and 22) see §10.15, §10.21 and §10.22.

From this we know that v_1^{10} is beyond only one facet. The extreme points that lie on this facet are points v^2, v^4, v^6 and v^7. The polytope $\text{conv}\{v^2, v^4, v^6, v^7, v_3^{10}\}$ is a 4-simplex with volume:

$$b_2a_3(b_1-a_1)^2(b_2-a_2)^2(b_3-a_3)^2/(24(b_2b_3-a_2a_3))\,.$$

The facets of $\text{conv}(\mathcal{P}_H \cup \{v_1^{10}\})$ are the facets of $\mathcal{P}_H$ except inequality 24. We see this by computing the four additional facets that come from adding v_1^{10} and noting that they are already contained in system H:

- The facet through points v^2, v^4, v^6 and v_1^{10} is 26.
- The facet through points v^2, v^4, v^7 and v_1^{10} is 33.
- The facet through points v^2, v^6, v^7 and v_1^{10} is 23.
- The facet through points v^4, v^6, v^7 and v_1^{10} is 32.

7.3. Computing the (additional) volume of conv($\mathcal{P}_H \cup \{v_1^{11}\} \cup \{v_1^{12}\}$). We now compute the additional volume of conv($\mathcal{P}_H \cup \{v_1^{11}\} \cup \{v_1^{12}\}$) compared to the volume of $\mathcal{P}_H$. Because $L(v_1^{11}, v_1^{12})$ lies entirely outside of $\mathcal{P}_1$, we need to add them sequentially.

We first compute the additional volume of conv($\mathcal{P}_H \cup \{v_1^{11}\}$) compared to the volume of $\mathcal{P}_H$. As we have done previously, we sum the volumes of conv($\{v_1^{11}\} \cup F$) for each facet, F, of $\mathcal{P}_H$ such that v_1^{11} is beyond that facet. We substitute v_1^{11} into each relevant inequality and if the result is negative then v_1^{11} lies beyond that facet. It is immediate that v_1^{11} satisfies inequalities 17, 18, 23-26 and 28-34. In §10.16 we show that 27 is also satisfied. It is also immediate that v_1^{11} violates the three facets described by 20-22. We compute the volume of the convex hulls of v_1^{11} with each of these facets.

The extreme points that lie on the first facet are points v^1, v^4, v^5 and v^7. The polytope conv$\{v^1, v^4, v^5, v^7, v_1^{11}\}$ is a 4-simplex with volume:

$$b_2a_3(b_1-a_1)^2(b_2-a_2)^2(b_3-a_3)^2/(24(b_2b_3-a_2a_3))\,.$$

The extreme points that lie on the second facet are points v^1, v^5, v^7 and v^8. The polytope conv$\{v^1, v^5, v^7, v^8, v_1^{11}\}$ is a 4-simplex with volume:

$$b_1a_3(b_1-a_1)(b_2-a_2)^3(b_3-a_3)^2/(24(b_2b_3-a_2a_3))\,.$$

The extreme points that lie on the third facet are points v^1, v^3, v^4 and v^5. The polytope conv$\{v^1, v^3, v^4, v^5, v_1^{11}\}$ is a 4-simplex with volume:

$$a_1b_2(b_1-a_1)(b_2-a_2)^2(b_3-a_3)^3/(24(b_2b_3-a_2a_3))\,.$$

Unlike in system 3, we see immediately that there exists a fourth facet (described by 19) which, under certain circumstances, v_1^{11} is beyond. In particular, this is true if and only if $a_1b_2b_3 - b_1a_2a_3 > 0$. Therefore we continue with two cases.

7.3.1. Case 1: $a_1b_2b_3 - b_1a_2a_3 > 0$ In this case there exists a fourth facet (described by 19) such that v_1^{11} is beyond this facet. The extreme points that lie on this fourth facet are points v^1, v^3, v^5 and v^8. The polytope conv$\{v^1, v^3, v^5, v^8, v_1^{11}\}$ is a 4-simplex with volume:

$$(a_1b_2b_3-b_1a_2a_3)(b_1-a_1)(b_2-a_2)^2(b_3-a_3)^2/(24(b_2b_3-a_2a_3))\,.$$

We now have a new polytope which is conv($\mathcal{P}_H \cup \{v_3^{11}\}$) (in case 1). We refer to this polytope as $\mathcal{T}_1^1$, and compute the facets of $\mathcal{T}_1^1$.

We begin with the facets of $\mathcal{P}_H$ and delete the four facets that v_1^{11} lies beyond (19-22). Let us call this system $\mathcal{T}_1^{1-}$. Now consider the four simplices we dealt with when computing the additional volume produced with v_1^{11}. Each of these simplices has 5 facets; one of which corresponds to a deleted facet of $\mathcal{P}_H$.

The remaining 4 facets of the first simplex are described by the planes through the following sets of points: $\{v^1, v^4, v^5, v_1^{11}\}$, $\{v^1, v^4, v^7, v_1^{11}\}$, $\{v^1, v^5, v^7, v_1^{11}\}$
and $\{v^4, v^5, v^7, v_1^{11}\}$.

The remaining 4 facets of the second simplex are described by the planes through the following sets of points: $\{v^1, v^5, v^7, v_1^{11}\}$, $\{v^1, v^5, v^8, v_1^{11}\}$, $\{v^1, v^7, v^8, v_1^{11}\}$
and $\{v^5, v^7, v^8, v_1^{11}\}$.

The remaining 4 facets of the third simplex are described by the planes through the following sets of points: $\{v^1, v^3, v^4, v_1^{11}\}$, $\{v^1, v^3, v^5, v_1^{11}\}$, $\{v^1, v^4, v^5, v_1^{11}\}$
and $\{v^3, v^4, v^5, v_1^{11}\}$.

The remaining 4 facets of the fourth simplex are described by the planes through the following sets of points: $\{v^1, v^3, v^5, v_1^{11}\}$, $\{v^1, v^3, v^8, v_1^{11}\}$, $\{v^1, v^5, v^8, v_1^{11}\}$ and $\{v^3, v^5, v^8, v_1^{11}\}$.

Consider these sixteen facets and exclude the facets that are shared by more than one simplex. This leaves eight facets.

We compute these eight facets to obtain the following:

- The facet through points v^1, v^3, v^8 and v_1^{11} is

$$\frac{1}{b_2b_3 - a_2a_3}\big(-a_1a_2^2a_3b_3 + a_1a_2b_2a_3b_3 + a_1a_2a_3b_3x_2 - a_1b_2a_3b_3x_2 - b_1a_2^2a_3^2 + b_1a_2^2a_3x_3 + a_2^2a_3b_3x_1 + b_1a_2a_3^2x_2 - b_1a_2a_3b_3x_2 + b_1a_2b_2b_3^2 - b_1a_2b_2b_3x_3 - a_2b_2b_3^2x_1 - a_2a_3f + b_2b_3f\big) \ge 0. \tag{52}$$

- The facet through points v^3, v^5, v^8 and v_1^{11} is

$$\frac{1}{b_2b_3 - a_2a_3}\big(-a_1a_2^2a_3b_3 + a_1a_2a_3b_3x_2 + a_1a_2b_2b_3x_3 + a_1b_2^2b_3^2 - a_1b_2^2b_3x_3 - a_1b_2b_3^2x_2 + a_2^2a_3b_3x_1 - b_1a_2b_2a_3b_3 + b_1a_2b_2a_3x_3 + b_1a_2b_2b_3^2 - b_1a_2b_2b_3x_3 - a_2b_2b_3^2x_1 - a_2a_3f + b_2b_3f\big) \ge 0. \tag{53}$$

- The facet through points v^1, v^4, v^7 and v_1^{11} is 33.
- The facet through points v^4, v^5, v^7 and v_1^{11} is 32.
- The facet through points v^1, v^7, v^8 and v_1^{11} is

$$f - b_2b_3x_1 - b_1a_3x_2 - b_1a_2x_3 + b_1a_2a_3 + b_1b_2b_3 \ge 0. \tag{54}$$

- The facet through points v^5, v^7, v^8 and v_1^{11} is 18.
- The facet through points v^1, v^3, v^4 and v_1^{11} is 17.
- The facet through points v^3, v^4, v^5 and v_1^{11} is

$$f - a_2a_3x_1 - a_1b_3x_2 - a_1b_2x_3 + a_1a_2a_3 + a_1b_2b_3 \ge 0. \tag{55}$$

There are four inequalities that are not already contained in system $\mathcal{T}_1^{1-}$, we add these and in doing so obtain the system of inequalities that describes $\mathcal{T}_1^1 = \mathrm{conv}(\mathcal{P}_H \cup \{v_1^{11}\})$ (in case 1).

We now compute the additional volume of $\mathrm{conv}(\mathcal{T}_1^1 \cup \{v_1^{12}\})$ compared to the volume of $\mathcal{T}_1^1$. As we have done previously, we sum the volumes of $\mathrm{conv}(\{v_1^{12}\} \cup F)$ for each facet, F, of $\mathcal{T}_1^1$ such that v_1^{12} is beyond that facet. We substitute v_1^{12} into each relevant inequality (i.e., the system that describes $\mathcal{T}_1^1$) and if the result is negative then v_1^{12} lies beyond that facet. It is immediately clear that v_1^{12} satisfies inequalities 17, 18, 23, 25-34 and 54-55. We also see immediately that inequalities 52 and 53 are violated. To see that inequality 24 is also satisfied see appendix §10.16.

Therefore we see that v_1^{12} is beyond two facets and we need to compute the volume of the convex hull of v_1^{12} with each of these facets.

The extreme points that lie on the first facet are points v^1, v^3, v^8 and v_1^{11}. The polytope $\mathrm{conv}\{v^1, v^3, v^8, v_1^{11}, v_1^{12}\}$ is a 4-simplex with volume:

$$a_2b_3(b_1 - a_1)^2(b_2 - a_2)^2(b_3 - a_3)^2/(24(b_2b_3 - a_2a_3)).$$

The extreme points that lie on the second facet are points v^3, v^5, v^8 and v_1^{11}. The polytope $\mathrm{conv}\{v^3, v^5, v^8, v_1^{11}, v_1^{12}\}$ is a 4-simplex with volume:

$$a_2b_3(b_1 - a_1)^2(b_2 - a_2)^2(b_3 - a_3)^2/(24(b_2b_3 - a_2a_3)).$$

We now compute the additional facets; we take the four facets from adding each simplex and delete the facet that repeats. This leaves us with the following six facet defining inequalities to compute:

- The facet through points v^1, v^3, v^8 and v_1^{12} is 31.
- The facet through points v^1, v^3, v_1^{11} and v_1^{12} is 17.
- The facet through points v^1, v^8, v_1^{11} and v_1^{12} is 54.
- The facet through points v^3, v^5, v^8 and v_1^{12} is 34.
- The facet through points v^3, v^5, v_1^{11} and v_1^{12} is 55.
- The facet through points v^5, v^8, v_1^{11} and v_1^{12} is 18.

By adding and deleting the appropriate facets from system H we see that we arrive at system 1.

Therefore, to compute the volume of $\mathcal{P}_1$, we sum the volume of $\mathcal{P}_H$ with that of the appropriate eight simplices, and we obtain our result for case 1.

7.3.2. Case 2: $a_1b_2b_3 - b_1a_2a_3 \leq 0$ In this case it is immediate to see that v_1^{11} satisfies 19 and therefore lies beyond no further facets. This means we now have a new polytope which is $\text{conv}(\mathcal{P}_H \cup \{v_3^{11}\})$ (in case 2). We refer to this polytope as $\mathcal{T}_1^2$, and we compute the facets of $\mathcal{T}_1^2$.

We begin with the facets of $\mathcal{P}_H$ and delete the three facets that v_1^{11} lies beyond (20-22). Let us call this system $\mathcal{T}_1^{2-}$. Now consider the four simplices we dealt with when computing the additional volume produced with v_1^{11}. Each of these simplices has 5 facets; one of which corresponds to a deleted facet of $\mathcal{P}_H$.

The remaining 4 facets of the first simplex are described by the planes through the following sets of points: $\{v^1, v^4, v^5, v_1^{11}\}$, $\{v^1, v^4, v^7, v_1^{11}\}$, $\{v^1, v^5, v^7, v_1^{11}\}$ and $\{v^4, v^5, v^7, v_1^{11}\}$.

The remaining 4 facets of the second simplex are described by the planes through the following sets of points: $\{v^1, v^5, v^7, v_1^{11}\}$, $\{v^1, v^5, v^8, v_1^{11}\}$, $\{v^1, v^7, v^8, v_1^{11}\}$ and $\{v^5, v^7, v^8, v_1^{11}\}$.

The remaining 4 facets of the third simplex are described by the planes through the following sets of points: $\{v^1, v^3, v^4, v_1^{11}\}$, $\{v^1, v^3, v^5, v_1^{11}\}$, $\{v^1, v^4, v^5, v_1^{11}\}$ and $\{v^3, v^4, v^5, v_1^{11}\}$.

Consider these twelve facets and exclude the facets that are shared by more than one simplex. This leaves eight facets.

We compute these eight facets to obtain the following:

- The facet through points v^1, v^4, v^7 and v_1^{11} is 33.
- The facet through points v^4, v^5, v^7 and v_1^{11} is 32.
- The facet through points v^1, v^5, v^8 and v_1^{11} is

$$\begin{aligned}&\frac{1}{b_2(b_1-a_1)}\big(-a_1b_1a_2^2a_3 + a_1b_1a_2a_3x_2 + a_1b_1a_2b_2x_3 - a_1b_1b_2^2b_3 + a_1b_2^2b_3x_1 + b_1a_2^2a_3x_1 \\ &\quad + b_1^2a_2b_2a_3 - b_1^2a_2a_3x_2 - b_1a_2b_2a_3x_1 + b_1^2a_2b_2b_3 - b_1^2a_2b_2x_3 - b_1a_2b_2b_3x_1 - a_1b_2f + b_1b_2f\big) \geq 0.\end{aligned} \tag{56}$$

- The facet through points v^1, v^7, v^8 and v_1^{11} is 54.
- The facet through points v^5, v^7, v^8 and v_1^{11} is 18.
- The facet through points v^1, v^3, v^4 and v_1^{11} is 17.
- The facet through points v^1, v^3, v^5 and v_1^{11} is

$$\begin{aligned}&\frac{1}{a_3(b_1-a_1)}\big(-a_1^2a_2a_3b_3 - a_1^2b_2a_3b_3 + a_1^2a_3b_3x_2 + a_1^2b_2b_3x_3 \\ &\quad + a_1b_1a_2a_3^2 + a_1a_2a_3b_3x_1 - a_1b_1a_3b_3x_2 + a_1b_2a_3b_3x_1 + a_1b_1b_2b_3^2 \\ &\quad - a_1b_1b_2b_3x_3 - a_1b_2b_3^2x_1 - b_1a_2a_3^2x_1 - a_1a_3f + b_1a_3f\big) \geq 0.\end{aligned} \tag{57}$$

- The facet through points v^3, v^4, v^5 and v_1^{11} is 55.

There are four inequalities that are not already contained in system $\mathcal{T}_1^{2-}$; we add these and in doing this we obtain the system of inequalities that describes $T_1^2 = \text{conv}(\mathcal{P}_H \cup \{v_1^{11}\})$ (in case 2).

We now compute the additional volume of $\text{conv}(\mathcal{T}_1^2 \cup \{v_1^{12}\})$ compared to the volume of $\mathcal{T}_1^2$. As we have done previously, we sum the volumes of $\text{conv}(\{v_1^{12}\} \cup F)$ for each facet, F, of $\mathcal{T}_1^2$ such that v_1^{12} is beyond that facet. We substitute v_1^{12} into each relevant inequality (i.e., the system of inequalities that describes T_1^2 in case 2) and if the result is negative then v_1^{12} lies beyond that facet. It is immediately clear that v_1^{12} satisfies inequalities 17, 18, 23, 25-34, 54 and 55. We also see immediately that v_1^{12} violates inequalities 19, 56 and 57. To see that 24 is also satisfied see §10.16.

Therefore we know that v_1^{12} is beyond three facets, and we need to compute the volume of the convex hull of v_1^{12} with each of these facets.

The extreme points that lie on the first facet are points v^1, v^3, v^5 and v^8. The polytope $\text{conv}\{v^1, v^3, v^5, v^8, v_1^{12}\}$ is a 4-simplex with volume:

$$a_2b_3(b_1 - a_1)^2(b_2 - a_2)^2(b_3 - a_3)^2/(24(b_2b_3 - a_2a_3))\,.$$

The extreme points that lie on the second facet are points v^1, v^5, v^8 and v_1^{11}. The polytope $\text{conv}\{v^1, v^5, v^8, v_1^{11}, v_1^{12}\}$ is a 4-simplex with volume:

$$(b_1a_2(b_1 - a_1)(b_2 - a_2)^2(b_3 - a_3)^3/(24(b_2b_3 - a_2a_3))\,.$$

The extreme points that lie on the third and final facet are points v^1, v^3, v^5 and v_1^{11}. The polytope $\text{conv}\{v^1, v^3, v^5, v_1^{11}, v_1^{12}\}$ is a 4-simplex with volume:

$$a_1b_3(b_1 - a_1)(b_2 - a_2)^3(b_3 - a_3)^2/(24(b_2b_3 - a_2a_3))\,.$$

We now compute the additional facets; we take the four facets from adding each simplex and delete the three facets that are repeated. This leaves us with the following six facet defining inequalities to compute:

- The facet through points v^1, v^3, v^8 and v_1^{12} is 31.
- The facet through points v^3, v^5, v^8 and v_1^{12} is 34.
- The facet through points v^1, v^8, v_1^{11} and v_1^{12} is 54.
- The facet through points v^5, v^8, v_1^{11} and v_1^{12} is 18.
- The facet through points v^1, v^3, v_1^{11} and v_1^{12} is 17.
- The facet through points v^3, v^5, v_1^{11} and v_1^{12} is 55.

By adding and deleting the appropriate facets from system H we see that we also arrive at system 1 in case 2.

Therefore, to compute the volume of $\mathcal{P}_1$, we sum the volume of $\mathcal{P}_H$ with that of the appropriate eight simplices, and we obtain our result for case 2. □

8. Proof of Thm. 4.3. A mapping from the proof of Theorem 4.4 allows us to claim Theorem 4.3 immediately. □

9. Possible extensions. Our results geometrically quantify the tradeoff between different convexifications of trilinear monomials. Of course it would be nice to use our results to develop guidelines for attacking trilinear monomials within an sBB code. In doing so, it should prove important to develop guidelines for how our results could be applied to formulations having many trilinear monomials overlapping on the same variables. We have seen that our results are very robust for scenarios where there is a high degree of overlap between trilinear monomials (see [27]). Also, we can imagine scoring each possible relaxation according to its volume, and then aggregating the scores to decide on what to do for each trilinear monomial. Another important issue is how

to effectively make branching decisions in the context of our relaxations. Guided by our volume results, we have made some significant progress in this direction (see [26]).

It would be natural and certainly difficult to extend our work to multilinear monomials having $n > 3$. In particular, advances for the important case of $n = 4$ could have immediate impact; [4] found, via experiments, that composing a trilinear and bilinear convexification in the manner suggested by $(x_i x_j) x_k x_l$ was a good strategy. They further observed sensitivity to the bounds on the variables, but they reached no clear conclusion on how to factor in that aspect. Restricting to this type of convexification, we could apply our results by substituting $w \in [a_i a_j, b_i b_j]$ to arrive at the trilinear monomial $w x_k x_l$, which can then be analyzed and relaxed according to our methodology. Of course, for a general quadrilinear monomial, there are six choices of which pair of variables will be treated as $\{x_i, x_j\}$, so we can analyze all six possibilities and take the best overall.

Also, there is the possibility of extending our results on trilinear monomials to (i) box domains that are not necessarily nonnegative, (ii) domains other than boxes, and (iii) other low-dimensional functions.

We hope that our work is just a first step in using volume to better understand and mathematically quantify the tradeoffs involved in developing sBB strategies for factorable formulations.

10. Appendix. Throughout the proofs, we have repeatedly claimed that certain quantities are nonnegative for any choice of $a_1, a_2, a_3, b_1, b_2, b_3$, such that, $0 < a_i < b_i$, for all i and

$$a_1 b_2 b_3 + b_1 a_2 a_3 \le b_1 a_2 b_3 + a_1 b_2 a_3 \le b_1 b_2 a_3 + a_1 a_2 b_3.$$

In this appendix, we provide proofs for the cases that are not immediate. As will become apparent, we need to demonstrate that many different 6-variable polynomials are nonnegative on the relevant parameter space. Generally, such demonstrations can be tricky global-optimization problems, and in many cases sum-of-squares proofs are not available; rather, we often make somewhat ad hoc arguments. Still, we can place some efficiency on all of this by establishing some technical lemmas.

10.1. We begin with the following lemmas that will be helpful in establishing the nonnegativity of certain quantities:

Lemma 10.1. *For all choices of parameters that meet our assumptions we have:* $b_1 a_2 - a_1 b_2 \ge 0$, $b_1 a_3 - a_1 b_3 \ge 0$ *and* $b_2 a_3 - a_2 b_3 \ge 0$.

Proof. $(b_3 - a_3)(b_1 a_2 - a_1 b_2) = b_1 a_2 b_3 + a_1 b_2 a_3 - a_1 b_2 b_3 - b_1 a_2 a_3 \ge 0$ by our original assumptions Ω. This implies $b_1 a_2 - a_1 b_2 \ge 0$, because $b_3 - a_3 > 0$. $b_1 a_3 - a_1 b_3 \ge 0$ and $b_2 a_3 - a_2 b_3 \ge 0$ follow from Ω in a similar way. □

Lemma 10.2. *Let* $A, B, C, D, E, F \in \mathbb{R}$ *with* $A \ge B \ge C \ge 0$ *and* $D \ge 0$, $E \ge 0$, $F \le 0$. *Also let,* $D + E + F = 0$. *Then* $AD + BE + CF \ge 0$.

Proof. $AD + BE + CF = AD + BE - C(D + E) \ge BD + BE - CD - CE = (B - C)(D + E) \ge 0$. □

Lemma 10.3. *Let* $A, B, C, D, E, F \in \mathbb{R}$ *with* $A \ge B \ge C \ge 0$ *and* $D \ge 0$, $E \le 0$, $F \le 0$. *Also let,* $D + E + F = 0$. *Then* $AD + BE + CF \ge 0$.

Proof. $AD + BE + CF = -A(E + F) + BE + CF = E(B - A) + F(C - A) \ge 0$. □

Lemma 10.4. *Let* $A, B, C, D \in \mathbb{R}$ *with* $A \ge B \ge 0$, $C + D \ge 0$, $C \ge 0$. *Then* $AC + BD \ge 0$.

Proof. $AC + BD \ge B(C + D) \ge 0$. □

10.2. Substituting $M(v_3^9, v_3^{11})$ into inequality 25 of the convex hull we obtain:

$$\begin{aligned}&\big(-a_1^2a_2^2a_3 + a_1^2a_2b_2b_3 + a_1a_2^2a_3b_1 + 2a_1a_2a_3b_1b_2 - 2a_1a_2b_1b_2b_3\\ &\quad -a_1a_3b_1b_2^2 - a_2^2a_3b_1^2 + a_2^2b_1^2b_3 - a_2b_1^2b_2b_3 + b_1^2b_2^2b_3\big) \Big/ 2(b_1b_2 - a_1a_2),\end{aligned}$$

the numerator of which can be rewritten as

$$b_1b_2\Big((b_1b_3 - a_1a_3)(b_2 - a_2)\Big) + b_1a_2\Big((b_1a_2 - a_1b_2)(b_3 - a_3)\Big) + a_1a_2\Big((b_2b_3 - a_2a_3)(a_1 - b_1)\Big),$$

and is nonnegative by Lemmas 10.1 and 10.2.

10.3. Substituting $M(v_3^{10}, v_3^{12})$ into inequality 26 of the convex hull we obtain:

$$\begin{aligned}&\big(-a_1^2a_2^2a_3 + a_1^2a_2a_3b_2 - a_1^2a_3b_2^2 + a_1^2b_2^2b_3 + a_1a_2^2b_1b_3 + 2a_1a_2a_3b_1b_2\\ &\quad -2a_1a_2b_1b_2b_3 - a_1b_1b_2^2b_3 - a_2a_3b_1^2b_2 + b_1^2b_2^2b_3\big) \Big/ 2(b_1b_2 - a_1a_2),\end{aligned}$$

the numerator of which can be rewritten as

$$b_1b_2\Big((b_2b_3 - a_2a_3)(b_1 - a_1)\Big) + a_1b_2\Big((b_1a_2 - a_1b_2)(a_3 - b_3)\Big) + a_1a_2\Big((b_1b_3 - a_1a_3)(a_2 - b_2)\Big),$$

and is nonnegative by Lemmas 10.1 and 10.3.

10.4. Substituting point v_3^{11} into inequality 25 of the convex hull or substituting v_3^{12} into inequality 26 we obtain:

$$\begin{aligned}&\big(-a_1^2a_2^2a_3 + a_1^2a_2b_2b_3 + a_1a_2^2b_1b_3 + 3a_1a_2a_3b_1b_2 - 3a_1a_2b_1b_2b_3\\ &\quad -a_1a_3b_1b_2^2 - a_2a_3b_1^2b_2 + b_1^2b_2^2b_3\big) \Big/ (b_1b_2 - a_1a_2).\end{aligned}$$

The numerator can be rewritten as

$$\begin{aligned}&b_3\Big(a_1a_2(a_1b_2 - b_1b_2) + b_1a_2(a_1a_2 - a_1b_2) + b_1b_2(b_1b_2 - a_1a_2)\Big)\\ &+ a_3\Big(a_1a_2(b_1b_2 - a_1a_2) + b_1a_2(a_1b_2 - b_1b_2) + b_1b_2(a_1a_2 - a_1b_2)\Big) \;=: b_3Y + a_3Z.\end{aligned}$$

Then we can see $Y + Z = (b_2 - a_2)(b_1 - a_1)(b_1b_2 - a_1a_2)$, which is nonnegative. Furthermore, by Lemma 10.3 we have $Y \geq 0$. Therefore, by Lemma 10.4 the numerator is nonnegative.

10.5. Substituting $M(v_1^9, v_1^{10})$ into inequality 19 of the convex hull we obtain:

$$\begin{aligned}&\big(-2a_1a_2^2a_3b_3 + a_1a_2^2b_3^2 - a_1a_2a_3^2b_2 + 4a_1a_2a_3b_2b_3 - a_1a_2b_2b_3^2 - a_1b_2^2b_3^2 + a_2^2a_3^2b_1\\ &\quad +a_2^2a_3b_1b_3 - a_2^2b_1b_3^2 - 4a_2a_3b_1b_2b_3 + 2a_2b_1b_2b_3^2 + a_3b_1b_2^2b_3\big) \Big/ 2(b_2b_3 - a_2a_3),\end{aligned}$$

the numerator of which can be rewritten as

$$\begin{aligned}&b_2b_3\Big(b_2(b_1a_3 - a_1b_3) + a_2(b_1b_3 - 2b_1a_3 + a_1a_3)\Big) + a_2b_3\Big((b_1 - a_1)(b_2 - a_2)(b_3 - a_3)\Big)\\ &+ a_2a_3\Big(b_2(2a_1b_3 - b_1b_3 - a_1a_3) + a_2(b_1a_3 - a_1b_3)\Big) \;=: b_2b_3X + a_2b_3Y + a_2a_3Z.\end{aligned}$$

Now, we write $X =: b_2V + a_2W$ and see that $V + W = (b_1 - a_1)(b_3 - a_3) \geq 0$ and, by Lemma 10.1, $V \geq 0$. Therefore $X \geq 0$ by Lemma 10.4. Because $X + Y + Z = 0$, by Lemma 10.2 we have that the numerator is nonnegative.

10.6. Substituting $M(v_1^9, v_1^{10})$ into inequality 20 of the convex hull, we obtain:

$$\big(-a_1a_2^2a_3b_3 - 2a_1a_2a_3^2b_2 + 4a_1a_2a_3b_2b_3 + a_1a_3^2b_2^2 - a_1a_3b_2^2b_3 - a_1b_2^2b_3^2 + a_2^2a_3^2b_1 \\ +a_2a_3^2b_1b_2 - 4a_2a_3b_1b_2b_3 + a_2b_1b_2b_3^2 - a_3^2b_1b_2^2 + 2a_3b_1b_2^2b_3\big) \Big/ 2(b_2b_3 - a_2a_3),$$

the numerator of which can be rewritten as

$$b_2b_3\Big(b_3(b_1a_2 - a_1b_2) + a_3(b_1b_2 - 2b_1a_2 + a_1a_2)\Big) + b_2a_3\Big((b_1 - a_1)(b_2 - a_2)(b_3 - a_3)\Big) \\ + a_2a_3\Big(b_3(2a_1b_2 - b_1b_2 - a_1a_2) + a_3(b_1a_2 - a_1b_2)\Big) \;=:\, b_2b_3X + b_2a_3Y + a_2a_3Z.$$

Now, we write $X =: b_3V + a_3W$ and see that $V + W = (b_1 - a_1)(b_2 - a_2) \geq 0$ and, by Lemma 10.1 $V \geq 0$. Therefore $X \geq 0$ by Lemma 10.4. Because $X + Y + Z = 0$, by Lemma 10.2 we have that the numerator is nonnegative.

10.7. Substituting $M(v_1^9, v_1^{11})$ into inequality 20 of the convex hull, we obtain:

$$\big(-a_1a_2^2a_3b_3 + 2a_1a_2a_3b_2b_3 - a_1a_3^2b_2^2 + a_1a_3b_2^2b_3 - a_1b_2^2b_3^2 + a_2^2a_3^2b_1 - a_2a_3^2b_1b_2 \\ -2a_2a_3b_1b_2b_3 + a_2b_1b_2b_3^2 + a_3^2b_1b_2^2\big) \Big/ 2(b_2b_3 - a_2a_3),$$

the numerator of which can be rewritten as

$$b_2b_3\Big((b_1a_2 - a_1b_2)(b_3 - a_3)\Big) + b_2a_3\Big((b_2a_3 - a_2b_3)(b_1 - a_1)\Big) + a_2a_3\Big((b_1a_3 - a_1b_3)(a_2 - b_2)\Big),$$

which is nonnegative by Lemmas 10.2 and 10.1.

10.8. Substituting $M(v_1^9, v_1^{11})$ into inequality 21 of the convex hull we obtain:

$$(b_2 - a_2)\left(a_1a_2a_3b_3 - a_1b_2b_3^2 - a_2a_3^2b_1 - a_2a_3b_1b_3 + a_2b_1b_3^2 + a_3^2b_1b_2\right) \Big/ 2(b_2b_3 - a_2a_3),$$

where the second multiplicand of the numerator can be rewritten as

$$b_3\Big(b_3(b_1a_2 - a_1b_2)\Big) + a_3\Big(a_1a_2b_3 - b_1a_2a_3 + b_1b_2a_3 - b_1a_2b_3\Big) \;=:\, b_3Y + a_3Z,$$

now $Y + Z = (b_1a_3 - a_1b_3)(b_2 - a_2) \geq 0$ (Lemma 10.1), and $Y \geq 0$ (Lemma 10.1), therefore by Lemma 10.4 we have that $b_3Y + a_3Z$ is nonnegative.

10.9. Substituting $M(v_1^9, v_1^{11})$ into inequality 22 of the convex hull we obtain:

$$(b_3 - a_3)\left(a_1a_2^2b_3 - a_1a_2b_2b_3 + a_1a_3b_2^2 - a_1b_2^2b_3 - a_2^2a_3b_1 + a_2b_1b_2b_3\right) \Big/ 2(b_2b_3 - a_2a_3),$$

where the second multiplicand of the numerator can be rewritten as

$$b_2\Big(b_3(b_1a_2 - a_1b_2) + a_1(b_2a_3 - a_2b_3)\Big) + a_2\Big(a_2(a_1b_3 - b_1a_3)\Big) \;=:\, b_2Y + a_2Z,$$

where $Y + Z = (b_1a_2 - a_1b_2)(b_3 - a_3) \geq 0$ and $Y \geq 0$ (both Lemma 10.1), therefore by Lemma 10.4 we have that this term is nonnegative.

10.10. Substituting $M(v_1^9, v_1^{11})$ into inequality 27 of the convex hull we obtain:

$$\begin{aligned}\big(&-a_1a_2^2a_3^2 + a_1a_2^2a_3b_3 - a_1a_2^2b_3^2 + 2a_1a_2a_3b_2b_3 - a_1a_3b_2^2b_3 + a_2^2b_1b_3^2 \\ &+a_2a_3^2b_1b_2 - 2a_2a_3b_1b_2b_3 - a_2b_1b_2b_3^2 + b_1b_2^2b_3^2\big) \Big/ 2(b_2b_3 - a_2a_3),\end{aligned}$$

the numerator of which can be rewritten as

$$b_2b_3\Big((b_1b_3 - a_1a_3)(b_2 - a_2)\Big) + a_2b_3\Big((b_2a_3 - a_2b_3)(a_1 - b_1)\Big) + a_2a_3\Big((b_1b_2 - a_1a_2)(a_3 - b_3)\Big),$$

which is nonnegative by Lemmas 10.3 and 10.1.

10.11. Substituting $M(v_1^{10}, v_1^{12})$ into inequality 19 of the convex hull we obtain:

$$\begin{aligned}\big(&-a_1a_2^2b_3^2 - a_1a_2a_3^2b_2 + 2a_1a_2a_3b_2b_3 + a_1a_2b_2b_3^2 - a_1b_2^2b_3^2 + a_2^2a_3^2b_1 \\ &-a_2^2a_3b_1b_3 + a_2^2b_1b_3^2 - 2a_2a_3b_1b_2b_3 + a_3b_1b_2^2b_3\big) \Big/ 2(b_2b_3 - a_2a_3),\end{aligned}$$

the numerator of which can be rewritten as

$$b_2b_3\Big((b_1a_3 - a_1b_3)(b_2 - a_2)\Big) + a_2b_3\Big((b_2a_3 - a_2b_3)(a_1 - b_1)\Big) + a_2a_3\Big((b_1a_2 - a_1b_2)(a_3 - b_3)\Big),$$

which is nonnegative by Lemma 10.3 and Lemma 10.1.

10.12. Substituting $M(v_1^{10}, v_1^{12})$ into inequality 21 of the convex hull we obtain:

$$(b_3 - a_3)\left(a_1a_2a_3b_2 - a_1b_2^2b_3 - a_2^2a_3b_1 + a_2^2b_1b_3 - a_2a_3b_1b_2 + a_3b_1b_2^2\right) \Big/ 2(b_2b_3 - a_2a_3),$$

where the second multiplicand of the numerator can be rewritten as

$$b_2\Big(b_2(b_1a_3 - a_1b_3)\Big) + a_2\Big(b_1(a_2b_3 - b_2a_3) + a_3(a_1b_2 - b_1a_2)\Big) \ =: b_2Y + a_2Z,$$

where $Y + Z = (b_1a_2 - a_1b_2)(b_3 - a_3) \geq 0$ and $Y \geq 0$ (both by Lemma 10.1). Therefore by Lemma 10.4 we have that $b_2Y + a_2Z$ is nonnegative.

10.13. Substituting $M(v_1^{10}, v_1^{12})$ into inequality 22 of the convex hull we obtain:

$$(b_2 - a_2)\left(a_1a_2b_3^2 + a_1a_3^2b_2 - a_1a_3b_2b_3 - a_1b_2b_3^2 - a_2a_3^2b_1 + a_3b_1b_2b_3\right) \Big/ 2(b_2b_3 - a_2a_3),$$

where the second multiplicand of the numerator can be rewritten as

$$b_3\Big(a_1(a_2b_3 - b_2a_3) + b_2(b_1a_3 - a_1b_3)\Big) + a_3\Big(a_3(a_1b_2 - b_1a_2)\Big) \ =: b_3Y + a_3Z,$$

where $Y + Z = (b_1a_3 - a_1b_3)(b_2 - a_2) \geq 0$ and $Z \leq 0 \Longrightarrow Y \geq 0$ (both by Lemma 10.1). Therefore by Lemma 10.4 we have that $b_3Y + a_3Z$ is nonnegative.

10.14. Substituting $M(v_1^{10}, v_1^{12})$ into inequality 24 of the convex hull we obtain:

$$\begin{aligned}\big(&-a_1a_2^2a_3^2 + a_1a_2a_3^2b_2 + 2a_1a_2a_3b_2b_3 - a_1a_2b_2b_3^2 - a_1a_3^2b_2^2 + a_2^2a_3b_1b_3 \\ &-2a_2a_3b_1b_2b_3 + a_3^2b_1b_2^2 - a_3b_1b_2^2b_3 + b_1b_2^2b_3^2\big) \Big/ 2(b_2b_3 - a_2a_3),\end{aligned}$$

the numerator of which simplifies to

$$b_2b_3\Big((b_1b_2 - a_1a_2)(b_3 - a_3)\Big) + b_2a_3\Big((b_2a_3 - a_2b_3)(b_1 - a_1)\Big) + a_2a_3\Big((b_1b_3 - a_1a_3)(a_2 - b_2)\Big),$$

and is nonnegative by Lemmas 10.1 and 10.2.

10.15. Substituting point v_1^9 into inequality 20 of the convex hull or substituting point v_1^{10} into inequality 19 we obtain:

$$\begin{aligned}&\big(-a_1a_2^2a_3b_3 - a_1a_2a_3^2b_2 + 3a_1a_2a_3b_2b_3 - a_1b_2^2b_3^2 + a_2^2a_3^2b_1\\ &\quad -3a_2a_3b_1b_2b_3 + a_2b_1b_2b_3^2 + a_3b_1b_2^2b_3\big)\Big/(b_2b_3 - a_2a_3),\end{aligned}$$

the numerator of which can be rewritten as

$$\begin{aligned}&b_3\Big(b_2(b_2(b_1a_3 - a_1b_3) + a_2(a_1a_3 + b_1b_3 - 2b_1a_3))\Big)\\ &+ a_3\Big(a_2(b_2(2a_1b_3 - a_1a_3 - b_1b_3) + a_2(b_1a_3 - a_1b_3))\Big) \ =: b_3Y + a_3Z.\end{aligned}$$

Now, we write $Y =: b_2^2V + a_2b_2W$ and see that $V + W = (b_1 - a_1)(b_3 - a_3) \geq 0$ and, by Lemma 10.1 $V \geq 0$. Therefore $Y \geq 0$ by Lemma 10.4. Because $Y + Z = (b_1a_3 - a_1b_3)(b_2 - a_2)^2 \geq 0$ (Lemma 10.1), by Lemma 10.4 we have that the numerator is nonnegative.

10.16. Substituting point v_1^{11} into inequality 27 of the convex hull or substituting point v_1^{12} into 24 we obtain:

$$\begin{aligned}&\big(-a_1a_2^2a_3^2 + 3a_1a_2a_3b_2b_3 - a_1a_2b_2b_3^2 - a_1a_3b_2^2b_3 + a_2^2a_3b_1b_3\\ &\quad +a_2a_3^2b_1b_2 - 3a_2a_3b_1b_2b_3 + b_1b_2^2b_3^2\big)\Big/(b_2b_3 - a_2a_3),\end{aligned}$$

the numerator of which can be rewritten as

$$\begin{aligned}&b_1\Big(b_2b_3(b_2b_3 - a_2a_3) + b_2a_3(a_2a_3 - a_2b_3) + a_2a_3(a_2b_3 - b_2b_3)\Big)\\ &+ a_1\Big(b_2b_3(a_2a_3 - a_2b_3) + b_2a_3(a_2b_3 - b_2b_3) + a_2a_3(b_2b_3 - a_2a_3)\Big) \ =: b_1Y + a_1Z,\end{aligned}$$

where: $Y + Z = (b_2b_3 - a_2a_3)(b_2 - a_2)(b_3 - a_3) \geq 0$, and by Lemma 10.3 we have $Y \geq 0$. Therefore by Lemma 10.4 we have that the numerator is nonnegative.

10.17. Substituting $M(v_1^9, v_1^{10})$ into inequality 21 of the convex hull we obtain

$$(2b_2b_3 - a_2b_3 - b_2a_3)(a_1a_2a_3 - a_1b_2b_3 - 2b_1a_2a_3 + b_1a_2b_3 + b_1b_2a_3)\Big/2(b_2b_3 - a_2a_3),$$

where the second multiplicand of the numerator can be rewritten as

$$b_2(b_1a_3 - a_1b_3) + a_2(a_1a_3 - 2b_1a_3 + b_1b_3),$$

which is nonnegative by Lemmas 10.1 and 10.4.

10.18. Substituting $M(v_1^9, v_1^{10})$ into inequality 22 of the convex hull we obtain

$$(a_2b_3 + b_2a_3 - 2a_2a_3)(a_1a_2b_3 + a_1b_2a_3 - 2a_1b_2b_3 - b_1a_2a_3 + b_1b_2b_3)\Big/2(b_2b_3 - a_2a_3),$$

where the second multiplicand of the numerator can be rewritten as

$$b_2(b_1b_3 - 2a_1b_3 + a_1a_3) + a_2(a_1b_3 - b_1a_3),$$

which is nonnegative by Lemmas 10.1 and 10.4.

10.19. Substituting point v_1^9 into inequality 21 of the convex hull we obtain

$$b_3(b_2 - a_2)\Big(b_2(b_1 a_3 - a_1 b_3) + a_2(a_1 a_3 - 2b_1 a_3 + b_1 b_3)\Big)\Big/(b_2 b_3 - a_2 a_3),$$

which is nonnegative by Lemmas 10.1 and 10.4.

10.20. Substituting point v_1^9 into inequality 22 of the convex hull we obtain

$$a_2(b_3 - a_3)\Big(b_2(b_1 b_3 - 2a_1 b_3 + a_1 a_3) + a_2(a_1 b_3 - b_1 a_3)\Big)\Big/(b_2 b_3 - a_2 a_3),$$

which is nonnegative by Lemmas 10.1 and 10.4.

10.21. Substituting point v_1^{10} into inequality 21 of the convex hull we obtain

$$b_2(b_3 - a_3)\Big(b_2(b_1 a_3 - a_1 b_3) + a_2(a_1 a_3 - 2b_1 a_3 + b_1 b_3)\Big)\Big/(b_2 b_3 - a_2 a_3),$$

which is nonnegative by Lemmas 10.1 and 10.4.

10.22. Substituting point v_1^{10} into inequality 22 of the convex hull we obtain

$$a_3(b_2 - a_2)\Big(b_2(b_1 b_3 - 2a_1 b_3 + a_1 a_3) + a_2(a_1 b_3 - b_1 a_3)\Big)\Big/(b_2 b_3 - a_2 a_3),$$

which is nonnegative by Lemmas 10.1 and 10.4.

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